ME-EE305

Matrix Algebra for Beginners[1] Part 1

Contents

[1] Gunawardena, Jeremy (2006). "Matrix algebra for beginners, Part I matrices, determinants, inverses". Department of Systems Biology Harvard Medical School 200 Longwood Avenue, Cambridge, MA 02115, USA.

Q1) What is a linear equation?

It is an equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$,
where $a_1, a_2, \ldots, a_n$ and b are constant real or complex numbers[2]. The linear refers to the fact that the unknown quantities appear just as x and y, not as 1/x or y [3]

Q2) What does "Systems of linear equations" mean?

A system of linear equations (or linear system) is a finite collection of linear equations in same variables. For instance, a linear system of m equations in n variables $x_1, x_2, \ldots, x_n$ can be written as:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Note: The set of all solutions of a linear system is called the solution set of the system.

Theorem 1

Any system of linear equations has one of the following exclusive conclusions: (a) No solution. (b) Unique solution. (c) Infinitely many solutions.

Q3) What is the main purpose of matrices?

Matrices first arose from trying to solve systems of linear equations.
A matrix is any rectangular array of numbers. If the array has **n** rows and **m** columns, then it is an **n × m** matrix.

Note: The numbers n and m are called the dimensions of the matrix

We will usually denote matrices with capital letters, like **A**, **B**, etc, although we will sometimes use lower case letters for one dimensional matrices (a vector) (ie: 1 × m or n × 1 matrices).

[2] A Complex Number is a combination of a Real Number and an Imaginary Number. Examples of real numbers: 1,12.38,−0.8625, ¾, √2. Example of imaginary number: when squared give a negative result. The imaginary number "i" is the square root of negative one. An imaginary number is a complex number that can be written as a real number multiplied by the imaginary unit i.

Note: One dimensional matrices are often called vector

Note: row vector is a n × 1 matrix

column vector is a 1 × m matrix

We will use the notation A_{ij} to refer to the number in the i-th row and j-th column.

Example of using matrices in Economics

Example 1: Two markets equilibrium Model

By Assuming perfectly competitive market: that is, both buyers and sellers are price-takers. Two goods (coffee and tea). Two goods are substitutable (not complementary). Each producer can produce only one good (short-run).

In **market-1**: we have the following equations:	In **market-2**: we have the following equations:
$Q_{d1} = 10 - 2P_1 + P_{2;}$ $Q_{s1} = -2 + 3P_1;$ $Q_{d1} = Q_{s1}$	$Q_{d2} = 15 + P_1 - P_{2;}$ $Q_{s2} = -1 + 2P_2;$ $Q_{d2} = Q_{s2}$
Demand for good 1, $Q_{d1} = f(P_1, P_2)$ The coefficient of P_1 is negative due to the law of demand. The coefficient of P_2 is positive due to the fact that the two goods are substitutes. Producer produces only good 1	Demand for good 2, $Q_{d2} = f(P_1, P_2)$ The coefficient of P_2 is negative due to the law of demand. The coefficient of P_1 is positive due to the fact that the two goods are substitutes. Producer produces only good 2

Note: As the number of equations increases, it becomes harder to solve a system of linear equations.

Question

How can we find the equilibrium prices and quantities for multiple market models?

Solution:

Step1: at equilibrium, equations can be rewritten as the following:

${Q_1}^* = 10 - 2P_1^* + P_2^*;$	${Q_2}^* = 15 + P_1^* - P_2^*;$
${Q_1}^* = -2 + 3P_1^*;$	${Q_2}^* = -1 + 2P_2^*;$

Step2: Using Matrices:

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix} * \begin{bmatrix} {Q_1}^* \\ {Q_2}^* \\ P_1^* \\ P_2^* \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \\ 15 \\ -1 \end{bmatrix}$$

All Coefficients of Endogenous Variables

All Constants

Or $AX = B$

Coefficient matrix * Endogenous Variable Vector= Constant Vector

The above matrix is a representation of the following system of linear equations:

$$1x{Q_1}^* + 0x{Q_2}^* + 2P_1^* - 1xP_2^* = 10 \quad \rightarrow \text{Eq (1)}$$
$$1x{Q_1}^* + 0x{Q_2}^* - 3P_1^* + 0xP_2^* = -2 \quad \rightarrow \text{Eq (2)}$$
$$0x{Q_1}^* + 1x{Q_2}^* - 1xP_1^* + 1xP_2^* = 15 \quad \rightarrow \text{Eq (3)}$$
$$0x{Q_1}^* + 1x{Q_2}^* + 0xP_1^* - 2P_2^* = -1 \quad \rightarrow \text{Eq (4)}$$

Step 3: The Solution

$$\begin{bmatrix} {Q_1}^* \\ {Q_2}^* \\ P_1^* \\ P_2^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 \end{bmatrix}^{-1} * \begin{bmatrix} 10 \\ -2 \\ 15 \\ -1 \end{bmatrix}$$

Or $X = A^{-1}B$

Endogenous Variable Vector= the inverse of Coefficient matrix * Constant Vector

Example 2: Expenditure model of national income

If you have the following variables for an economy: Y = Income C = Consumption I = Investment G = Government expenditure.
If you know that income is the summation of all expenditures by the government, the household, and the private sector; the consumption function is a function of income provided that there exists an autonomous consumption. If you know that consumption and income are endogenous; while investment and government expenditure are exogenous.

Question
What are the values of income and consumption?

Step1:
Given the previous information, we can find that:

Y = C + I + G ➔ **Income Equation**

C = a + bY ➔ **Consumption Equation**

Step2:
Using Matrices:

$$\begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix} * \begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} I + G \\ a \end{bmatrix}$$

All Coefficients of Endogenous Variables

All Exogenous Variables or Constants

$$\textbf{Or } \boldsymbol{AX = B}$$

Coefficient matrix * Endogenous Variable Vector= Constant Vector

The above matrix is a representation of the following system of linear equations:

1xY - 1xC = I + G ➔ **Eq (1)**
-bY + 1xC = a ➔ **Eq (2)**

Step 3: The Solution

$$\begin{bmatrix} Y \\ C \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -b & 1 \end{bmatrix}^{-1} * \begin{bmatrix} I + G \\ a \end{bmatrix}$$

Or $\boldsymbol{X = A^{-1}B}$

Endogenous Variable Vector= the inverse of Coefficient matrix * Constant Vector