Rule ①

* derivative of the exponential Functions: $y = e^{f(t)}$

$: y = e^{x}, x = f(t)$

$$\frac{d}{dt} e^{f(t)} = f'(t) \cdot e^{f(t)}$$

Rule ②

* derivative of the logarithmic Function: ~~y = lnf(t)~~ $y = \ln f(t)$

$$\frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)}$$

Rule ③ instantaneous rate of growth

$$r_y \equiv \frac{dy/dt}{y} = \frac{\text{Marginal function}}{\text{Total function}}$$

USIC1 instantaneous rate of change: the rate of change at a particular moment

a) find the derivatives of the following functions: → with respect to (t)

① $y = e^{rt}$ ④ $y = \ln t^{c}$

② $y = e^{-t}$

③ $y = \ln(at)$

Solutions

① $\frac{dy}{dt} = \frac{d}{dt} e^{rt} = r \cdot e^{rt}$

② $\frac{dy}{dt} = \frac{d}{dt} e^{-t} = -e^{-t}$

③ $\frac{dy}{dt} = \frac{d}{dt} \ln(at) = \frac{1}{at} \cdot a = \frac{1}{t}$

④ $\frac{dy}{dt} = \frac{d}{dt} t^{c} = \frac{1}{t^{c}} \cdot c t^{c-1} = \frac{c}{t}$

Q2) Find the first Four derivatives of the rational function using the product rule

$$y = g(x) = \frac{x}{1+x} \quad , (x \neq -1)$$

Solution

$$\left.\begin{aligned} g(x) &= x(1+x)^{-1} \\ g'(x) &= (1+x)^{-2} \\ g''(x) &= -2(1+x)^{-3} \\ g'''(x) &= 6(1+x)^{-4} \\ g''''(x) &= -24(1+x)^{-5} \end{aligned}\right\} x \neq -1$$

Q) Find the rate of growth of
$v = A e^{rt}$, where (t) denotes Time.

$\boxed{\text{Solution}}$

* The rate of growth of (v) is r.

$\boxed{\text{Prove}}$

b) Taking (ln)

$$\ln v = \ln A + rt \ln e$$
$$= \ln A + rt$$

Note (A) is constant

$$\frac{d \ln v}{dt} = 0 + \frac{d}{dt}(rt) = r$$

Q) Find the rate of growth of
2

$y = 4^t$

$\ln y = \ln 4^t = t \ln 4$

$\frac{d \ln y}{dt} = \ln 4$

Summary

rate of growth

When a variable (y) is a function of time, $y = f(t)$, its instantaneous rate of growth is defined as:

$$r_y = \frac{dy/dt}{y} = \frac{f'(t)}{f(t)} = \frac{\text{marginal function}}{\text{Total function}}$$

* Note: $\frac{f'(t)}{f(t)} = \frac{d \ln f(t)}{dt}$

* Finding Extreme (Maximum/ Minimum) Points For an objective function in the case of more than one choice variable.

1) First-order Condition

if $Z = f(x)$

$dZ = f'(x) \cdot dx = \text{zero}$

"the first order differential condition"

$\boxed{\text{remember}}$

↳ while the first-order differential condition is necessary for an extremum, it is not sufficient. since, an inflection point can also satisfy the condition that $dZ = 0$.

2) Second-order Condition   * differential equivalent.

$f''(x) < 0$ (Maximum) $\rightarrow d^2Z \leq 0$

$f''(x) < 0$ (Minimum) $\rightarrow d^2Z \geq 0$

that is, $d^2Z \equiv d(dZ) = d[f'(x)dx]$

$= f''(x)\, dx^2$

Difference between differential & derivative

↳ both terms refer to interrelationship

↳ derivative refers to the rate of change of one variable with respect to another

→ equations which define relationship between ~~these~~ variables and their derivatives are called differential equations.

→ differentiation: is the process of finding derivatives.

~~the~~

[briefly speaking]

the derivative of a function is the rate of change of the output value with respect to its input value, whereas differential is the actual change of the function.

* differentials are represented as:

dx, dy, dz

where

dx : represents a small change in x
dz : " " " " z
dt : " " " " t.

that is:

$$\boxed{dy = f'(x).dx}$$

we usually add the Condition
" For arbitrary values of $dx \neq 0$ "

[X] Extreme Values of a function of Two Variables

let's say $z = f(x,y)$ "Two independent Variables"

→ First order Condition.

$$dz = \text{zero}$$

" For arbitrary values of dx & $dy \neq 0$ "

that is, the total differential is

$$dz = f_x\, dx + f_y\, dy = \text{zero}$$

this implies that:

$$f_x = f_y = \text{zero}$$

or

$$\left[\frac{\delta z}{\delta x} = \frac{\delta z}{\delta y} = 0\right]$$

the equivalent derivative version of the first-order Condition

* Second order Condition

→ $d^2z \leq 0$ # for arbitrary values of dx & dy not equal to zero.

↳ Maximum

→ $d^2z \geq 0$ " for arbitrary values of dx & dy not equal to zero "

↳ Minimum

Note that:

$d^2z < 0$ iff $f_{xx} < 0$; $f_{yy} < 0$; $f_{xx} f_{yy} > f_{xy}^2$

$d^2z > 0$ iff $f_{xx} > 0$; $f_{yy} > 0$; $f_{xx} f_{yy} > f_{xy}^2$

Condition	Maximum	Minimum
First order necessary	$f_x = f_y = 0$	$f_x = f_y = 0$
Second order sufficient	$f_{xx} < 0$ $f_{yy} < 0$ $f_{xx} f_{yy} > f_{xy}^2$	$f_{xx} > 0$ $f_{yy} > 0$ $f_{xx} f_{yy} > f_{xy}^2$

Q) Find the extreme values of the Following Function:

$$Z = x + 2ey - e^x - e^{2y}$$

Solution

$f_x = 1 - e^x$

$f_{xx} = -e^x$

$f_{xy} = zero$

$f_y = 2e - 2e^{2y}$

$f_{yy} = -4e^{2y}$

$f_{yx} = zero$

* 1st order Condition:

$1 - e^x = zero$

$2e - 2e^{2y} = zero$

that is $x^* = 0$, $y^* = \frac{1}{2}$

* 2nd order Condition:

at $x^* = 0$ & $y^* = \frac{1}{2}$

$$\left.\begin{array}{l} f_{xx} = -1 < 0 \\ f_{yy} = -4e < 0 \\ f^2_{xy} = zero \\ f_{xx} f_{yy} = 4e > f^2_{xy} \end{array}\right\} \text{Maximum}$$

stationary Points

$$(\underset{x^*}{0}, \underset{y^*}{\tfrac{1}{2}}, \underset{z^*}{-1})$$

* Hessian Matrix "determinant"

it is a square matrix of second-order partial derivatives.

$$H = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 x_2} & \frac{\delta^2 f}{\delta x_1 x_3} \\ \frac{\delta^2 f}{\delta x_2 x_1} & \frac{\delta^2 f}{\delta x_2^2} & \frac{\delta^2 f}{\delta x_2 x_3} \\ \frac{\delta^2 f}{\delta x_3 x_1} & \frac{\delta^2 f}{\delta x_3 x_2} & \frac{\delta^2 f}{\delta x_3^2} \end{bmatrix}_{3\times 3}$$

$$H = \begin{bmatrix} \frac{\delta^2 f}{\delta x_1^2} & \frac{\delta^2 f}{\delta x_1 x_2} \\ \frac{\delta^2 f}{\delta x_2 x_1} & \frac{\delta^2 f}{\delta x_2^2} \end{bmatrix}_{2.2}$$

the Hessian determinant : $|H|$

it is also called "the discriminant"

it should be noted that:

d^2z is positive definite iff

$f_{xx} > 0$ & $|H| > 0$

d^2z is negative definite iff

$f_{xx} < 0$ & $|H| > 0$

(5) given that $f_{xx} = -2$, $f_{xy} = 1$

$f_{yy} = -1$ at a certain point on a function

$z = f(x, y)$

does d^2z have a definite sign at that point regardless of the values of dx & dy?

$\boxed{\text{Solution}}$

the discriminant of the quadratic form d^2z

$$\boxed{-2 < 0} \ \& \ \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} = 1 > 0$$

thus d^2z is negative definite

* the Second order Condition using the Hessian determinant:

2nd order Condition

$$|H| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

the leading Principal Minors are:

$$H_1 = |f_{11}|$$

$$H_2 = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$H_3 = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$\boxed{\text{Maximum}}$ negative definite

if $|H_1| < 0$

& $|H_2| > 0$

& $|H_3| < 0$

$\boxed{\text{Minimum}}$ Positive definite

if $|H_1| > 0$

& $|H_2| > 0$

& $|H_3| > 0$

(5) Find the relative extremum of the average - cost functions

$$AC = f(Q) = Q^2 - 5Q + 8$$

$\boxed{\text{Solution}}$

$f'(Q) = 2Q - 5$

↳ a linear function

$f'(Q) = \text{zero}$

then

$Q^* = 2.5$, $AC = 1.75$

critical values are

$f''(Q) = 2 > 0$

$\boxed{\text{a relative Minimum}}$

Q) Find the extreme values of

$$Z = 2x_1^2 + x_1 x_2 + 4x_2^2 + x_1 x_3 + x_3^2 + 2$$

Solution

* 1st order Condition:-

$$f_1 = f_2 = f_3 = 0$$

$$f_1 = 4x_1 + x_2 + x_3 = 0$$

$$f_2 = x_1 + 8x_2 = 0$$

$$f_3 = x_1 + 2x_3 = 0$$

$$x_1^* = x_2^* = x_3^* = 0$$

stationary points are $(0,0,0,2)$

* 2nd order Condition

$$H = \begin{vmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

& $|H_1| = 4 > 0$

$|H_2| = 31 > 0$

$|H_3| = 54 > 0$

We Can Conclude that $Z^* = 2$ "Minimum"

(Case of two choice variables & two price parameters)

(Perfect Competition)

Q) if firm (A) is a two product firm working under pure competition. the firm's revenue function is given by:

$$R_1 = P_{10} Q_1 + P_{20} Q_2$$

the firm's Cost Function is assumed to be:

$$C = 2Q_1^2 + Q_1 Q_2 + 2Q_2^2$$

given $P_{10} = 12$, $P_{20} = 18$

→ Find the Level of Q_1 & Q_2 that will maximize profits.

* the Profit Function: Note: P_{10} & P_{20} are exogenous

$$\pi = R - C = P_{10} Q_1 + P_{20} Q_2 - 2Q_1^2 - Q_1 Q_2 - 2Q_2^2$$

* 1st order Conditions:

$$\frac{\delta \pi}{\delta Q_1} = P_{10} - 4Q_1 - Q_2 \qquad \frac{\delta \pi}{\delta Q_2} = P_{20} - Q_1 - 4Q_2$$

then $\boxed{4Q_1 + Q_2 = P_{10}}$ $\boxed{Q_1 + 4Q_2 = P_{20}}$

by Solving the two Simultaneous equations:

$$Q_1^* = \frac{4P_{10} - P_{20}}{15} \quad \& \quad Q_2^* = \frac{4P_{20} - P_{10}}{15}$$

$$\boxed{Q_1^* = 2} \quad \& \quad \boxed{Q_2^* = 4} \quad \& \quad \boxed{\pi^* = 48}$$

⟹

$\boxed{\text{*2nd order Condition}}$

$$|H| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix} = \begin{vmatrix} -4 & -1 \\ -1 & -4 \end{vmatrix}$$

$$\left.\begin{array}{l} |H_1| = -4 < 0 \\ |H_2| = 15 > 0 \end{array}\right\} \text{negative definite}$$

Maximum

$\boxed{\text{Note}}$ Since the signs of the leading principal minors do not depend on where they are evaluated d^2z is everywhere negative definite. "strictly concave"

↓

in this case, we have a unique absolute maximum.

The Case of two choice variables & Two price Parameters "Monopoly"

Q) if firm (B) is a Two Product Firm working under a monopolist market', the firm's demand Functions are given by:-

$$Q_1 = 40 - 2P_1 + P_2 \longrightarrow (1)$$
$$Q_2 = 15 + P_1 - P_2 \longrightarrow (2)$$

the Cost Function is given by: $\boxed{C = Q_1^2 + Q_1 Q_2 + Q_2^2}$

a) what are the relationship between The Two goods??

b) Find the average revenue Functions.

c) Find The output levels that will maximize Profit?

a) the two goods are substitutes, because an increase in the price of one will raise the demand for the other.

b) $\boxed{\text{average revenue functions}}$ $\boxed{P_1 \equiv AR_1}$ & $\boxed{P_2 \equiv AR_2}$

From (1) & (2)

$$-2P_1 + P_2 = Q_1 - 40$$
$$P_1 - P_2 = Q_2 - 15$$

by using Cramer's rule to Solve P_1 & P_2

$$P_1 = \frac{\begin{vmatrix} Q_1 - 40 & 1 \\ Q_2 - 15 & -1 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = 55 - Q_1 - Q_2$$

$$P_2 = \frac{\begin{vmatrix} -2 & Q_1 - 40 \\ 1 & Q_2 - 15 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = \frac{\begin{vmatrix} -2 & Q_1 - 40 \\ 1 & Q_2 - 15 \end{vmatrix}}{\begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}} = 70 - Q_1 - 2Q_2$$

E2 the Firms Total revenue:

$$R = P_1Q_1 + P_2Q_2 = 55Q_1 + 70Q_2 - 2Q_1Q_2 - Q_1^2 - 2Q_2^2$$

then the Profit Function will be:-

$$\pi = R - C$$

$$= \boxed{55Q_1 + 70Q_2 - 3Q_1Q_2 - 2Q_1^2 - 3Q_2^2}$$

the objective function with two choice variables:-

$$\pi_1' = 55 - 3Q_2 - 4Q_1 \quad \& \quad \pi_2' = 70 - 3Q_1 - 6Q_2$$

$$\pi_{11}'' = \cancel{5}-4 \qquad , \pi_{22} = -6 \qquad \pi_{12} = -3$$

$\boxed{\text{1st order Condition}}$

$$\begin{aligned} 4Q_1 + 3Q_2 &= 55 \\ 3Q_1 + 6Q_2 &= 70 \end{aligned} \Rightarrow (Q_1^*, Q_2^*) = (8, 7{,}67)$$

$$P_1^* = 39{,}3 \qquad P_2^* = 4{,}67 \qquad \pi^* = 488{,}3$$

$\boxed{\text{2nd order Condition}}$

$$|H| = \begin{vmatrix} -4 & -3 \\ -3 & -6 \end{vmatrix}$$

$$\left.\begin{aligned} |H_1| &= -4 < 0 \\ |H_2| &= 15 > 0 \end{aligned}\right\} \text{negative definite "everywhere"}$$

strictly concave

unique absolute maximum

Case of Monopolist "Price discrimination"

Suppose that a monopolist Company (C) has the following " average - revenue functions":

$$P_1 = 63 - 4Q_1 \rightarrow R_1 = 63Q_1 - 4Q_1^2$$
$$P_2 = 105 - 5Q_2 \rightarrow R_2 = 105Q_2 - 5Q_2^2$$
$$P_3 = 75 - 6Q_3 \rightarrow R_3 = 75Q_3 - 6Q_3^2$$

if the total Cost Function is given by:

$$C = 20 + 15Q$$

Q) Find the equilibrium quantities & prices

$\boxed{\text{Solution}}$

$$R_1' = 63 - 8Q_1$$
$$R_2' = 105 - 10Q_2 \quad , \quad C' = 15$$
$$R_3' = 75 - 12Q_3$$

$\boxed{R_1' = MC}$ & $\boxed{R_2' = MC}$, $\boxed{R_3' = MC}$

Setting $R_1' = \cancel{R_2' = R_3'} = MC$, we can find:

$\boxed{Q_1^* = 6}$ & $\boxed{Q_2^* = 9}$ & $\boxed{Q_3^* = 5}$

$\boxed{\pi^* = 679}$

$\boxed{P_1^* = 39}$, $\boxed{P_2^* = 60}$, $\boxed{P_3^* = 45}$

~~* 2nd order Condition~~

$\boxed{\text{Note}}$ the Point elasticity of demand is Lowest in the Second market, in which the highest Price is charged.

* 2nd order condition is "Satisfied" "Maximum"

$$Q_1 = 15.75 - 0.25(P_1) \rightarrow \frac{\delta Q_1}{\delta P_1} = -0.25 P_1$$

$$Q_2 = 21 - 0.2(P_2) \rightarrow \frac{\delta Q_2}{\delta P_2} = -0.2 P_2$$

$$Q_3 = 12.5 - 0.16(P_3) \rightarrow \frac{\delta Q_3}{\delta P_3} = -0.16 P_3$$

$$E_1 = -0.25 \cdot \frac{P_1^*}{Q_1^*} = -1.625$$

$$E_2 = -0.2 \frac{P_2^*}{Q_2^*} = \boxed{-1.34}$$ ← lowest elasticity

$$E_3 = 0.16 \frac{P_3^*}{Q_3^*} = -1.44$$

Labor & Capital "optimal choice of inputs"

if you have ~~the foll~~ Firm (D) working under a Competitive environment with the following ~~Profit~~ Revenue Function & Cost Function

~~$\pi = R - C = PQ - wL - rK$~~

$R = PQ$; $Q = L^{\alpha} K^{\beta}$ "Cobb-douglas Function"

$C = wL + rK$; $Q = F(L, K)$

where P (Price); Q (output); L (Labor); K (Capital)

w, r input prices for L & K

Q) a) what are the main exogenous variables? $\boxed{P, w \& r}$

b) what are the main endogenous variables? $\boxed{K, L, Q}$

c) what are the optimal levels of L & K that maximize $\boxed{\pi}$?

when $\alpha = \beta$

Solution

$$\pi = P \times Q - C$$

$$\pi = P L^{\alpha} K^{\alpha} - wL - rK$$

1st order Condition

$$\frac{\delta \pi}{\delta L} = P\alpha L^{\alpha-1} K^{\alpha} - w = 0 \longrightarrow (1)$$

$$\frac{\delta \pi}{\delta K} = P\alpha L^{\alpha} K^{\alpha-1} - r = 0 \longrightarrow (2)$$

From ① $P\alpha L^{\alpha-1} K^{\alpha} = w$

$$K = \left(\frac{w}{P\alpha} L^{1-\alpha}\right)^{\frac{1}{\alpha}}$$

Substituting into ②

$$P\alpha L^{\alpha} K^{\alpha-1} - r = P\alpha L^{\alpha} \left[\left(\frac{w}{P\alpha} L^{1-\alpha}\right)^{\frac{1}{\alpha}}\right]^{\alpha-1} - r = 0$$

$$r = P^{\frac{1}{\alpha}} \alpha^{\frac{1}{\alpha}} w^{(\alpha-1)/\alpha} L^{(2\alpha-1)/\alpha}$$

$$L^{*} = (P\alpha w^{\alpha-1} r^{-\alpha})^{\frac{1}{(1-2\alpha)}}$$

$$K^{*} = (P\alpha r^{\alpha-1} w^{-\alpha})^{\frac{1}{(1-2\alpha)}}$$

We can substitute L^* & K^* into the production function to get (Q^*)

$$Q^* = \left(\frac{\alpha^2 P^2}{wr}\right)^{\alpha/(1-2\alpha)}$$