Leaflet- Optimum Values and Extreme Values

Contents

Introduction

Economics is usually a science of choice. When an economic project is to be carried out, such as the production of a specific level of output, there are normally a number of alternative ways of accomplishing it. One or more of these alternatives will, however, be more desirable than other from standpoint of some creation and it is essence of optimization problem to choose, on the basic off that specified criterion the best alternative available.

The most common criterion of choice among alternatives in economics is the goals of maximizing something (such as maximizing a firm's profits, a consumer's utility or a rate of growth of a firm or of a country's economy or of minimizing something (such as minimizing the cost of producing a given output).Economically we may categorize such maximization and minimization problem under the general heading of optimization meaning "the quest for the best".

For example,

A business firm seeks to maximize profits: $\pi(Q) = R(Q) - C(Q)$

This equation constitutes the relevant **objective function** with π as the objective of maximization and Q as the choice variable.

The optimization problem is represented as choosing the level of Q that maximizes π.

Note that while the optimal level of π is by definition is <u>maximal level</u>, the optimal level of choice variable Q is itself not required to be either a maximum of a minimum.

Key Difference:

- **'Optimum'** means the quality, quantity or degree of something which is best or most favorable. Optimum means using or producing something to the best level.
- **'Maximum'** can be defined as the highest, largest or biggest possible of something. Maximum means using or producing something to the highest level.
- **'Minimum'** can be defined as the smallest value of a set, function, etc. Minimum means using or producing something to the lowest cost.

What is Objective function?

The function that it is to be maximized or minimized. A function to be optimized given certain constraints and with variables that need to be minimized or maximized. That function includes two types of variables:

- **Dependent variable:** to be maximized or minimized
- **Explanatory variables:** choice/ decision/ policy variables

It should be noted that e**xtreme values** are for the **dependent** variable in the objective function and <u>**not**</u> for the **choice** variables.

Finding the Relative Maximum and Minimum

Definitions:

A Critical Point (stationary point): a point x_0 in the domain at which either the derivative is zero or it does not exist. geometrically, one of the following cases happen at a critical point:

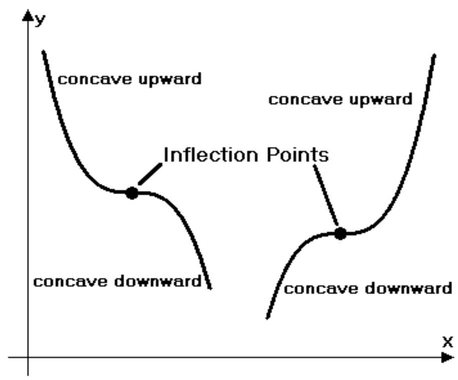
- The tangent line is horizontal
- The tangent line is vertical (this corresponds to the case where the derivative is one of $\pm\infty$)

- The tangent line does not exist; there is a cusp on the graph at x_0. Note: (A cusp is a point at which two branches of a curve meet such that the tangents of each branch are equal).

Any point at which the tangent to the graph is horizontal is called a critical point or stationary point. We can locate stationary points by looking for points at which $\frac{dy}{dx} = 0$.

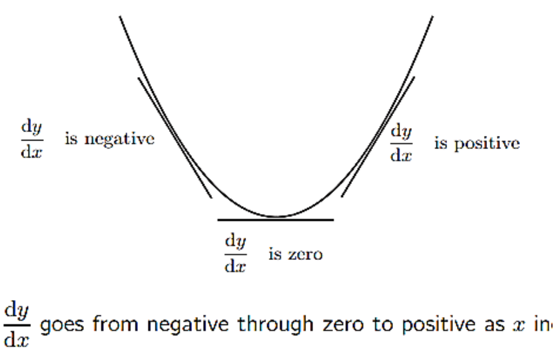
Inflection Point

A point of a curve at which the curve changes from being concave to convex, or vice versa.



Case 1: Local or relative Minimum

A Relative Minimum: It occurs if the derivative $f'(x)$ changes its sign from negative to positive from the immediate left of the point x_0 to its immediate right.



Notice that to the left of the minimum point, $\frac{dy}{dx}$ is negative because the tangent has negative gradient. At the minimum point, $\frac{dy}{dx}$ = 0. To the right of the minimum point $\frac{dy}{dx}$ is positive, because here the tangent has a positive gradient. So, $\frac{dy}{dx}$ goes from negative, to zero, to positive as x increases. In other words, $\frac{dy}{dx}$ must be increasing as x increases.

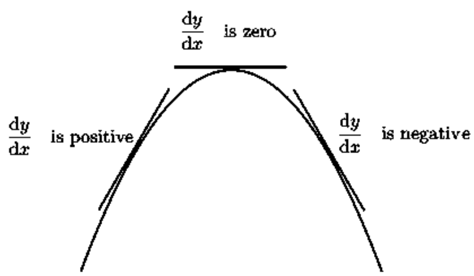
Note: It is important to realize that this test for a maximum is not conclusive. It is possible for a stationary point to be a minimum even if $\frac{d^2y}{dx^2} = 0$,

Note: (the stationary point is a minimum → If $\frac{dy}{dx}$ is increasing near the stationary point then that point must be minimum (a minimum turning point). if the derivative of $\frac{dy}{dx}$ is positive then we will know that $\frac{dy}{dx}$ is increasing. That is, if the second derivative (i.e., $\frac{d^2y}{dx^2}$ is positive).

Rule: if $\frac{dy}{dx} = 0$ at a point, and if $\frac{d^2y}{dx^2} > 0$, then that point must be a minimum.

Case 2: Local or Relative Maximum

A relative maximum: It occurs if the derivative $f'(x)$ changes its sign from positive to negative from the immediate left of the point x_0 to its immediate right.



$\frac{dy}{dx}$ goes from positive through zero to negative as x increases.

Notice that to the left of the maximum point, $\frac{dy}{dx}$ is positive because the tangent has positive gradient. At the maximum point, $\frac{dy}{dx} = 0$. To the right of the maximum point $\frac{dy}{dx}$ is negative, because here the tangent has a negative gradient. So, $\frac{dy}{dx}$ goes from positive, to zero, to negative as x increases. In other words, $\frac{dy}{dx}$ must be decreasing as x increases.

Note: (the stationary point is a maximum → If $\frac{dy}{dx}$ is decreasing near the stationary point then that point must be maximum (a maximum turning point). if the derivative of $\frac{dy}{dx}$ is negative then we will know that $\frac{dy}{dx}$ is decreasing. That is, if the second derivative (i.e., $\frac{d^2y}{dx^2}$ is negative).

Note: It is important to realize that this test for a maximum is not conclusive. It is possible for a stationary point to be a maximum even if $\frac{d^2y}{dx^2} = 0$,

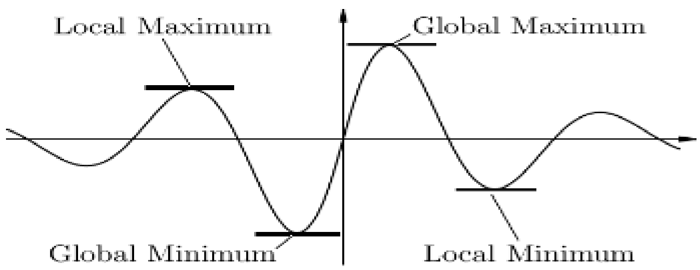
Rule: if $\frac{dy}{dx} = 0$ at a point, and if $\frac{d^2y}{dx^2} < 0$, then that point must be a maximum.

Case 3: Neither a relative Maximum nor relative Minimum

Neither a relative Maximum nor relative Minimum: It occurs if the derivative $f'(x)$ has the same sign on both the immediate left and the immediate right of point x_0. Example, the case of Inflection Point.

A graphical representation of relative/ local extreme points vs. absolute/ global extreme points.

The local and global extremes



Note: The gradient of this graph is zero at each of the horizontal tangent lines. the gradient is given by $\frac{dy}{dx}$ and it should equal zero. All of these points are known as stationary points.

Summary

The First Derivative Test (The Necessary Condition)**:**

- If $\frac{dy}{dx} = 0$, then we can state that the points we are looking for are critical values.

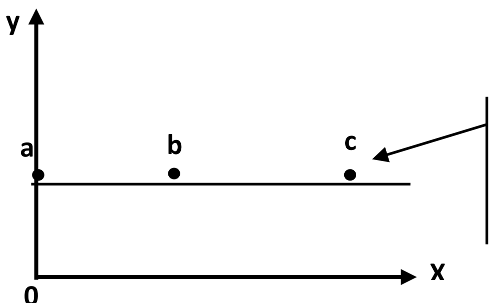
Note: It is possible that some such points will not be turning points.
To apply the first-derivative test, we need to test values for $x \gtreqless \textit{the critical values}$.

The Second Derivative Test

- If $\frac{d^2y}{dx^2} > 0$, then the stationary point is a minimum turning point **(the Sufficient Condition)**.
- If $\frac{d^2y}{dx^2} < 0$, then the stationary point is a maximum turning point **(the Sufficient Condition)**.
- If $\frac{d^2y}{dx^2} = 0$, it is possible that we have a maximum, or a minimum, or indeed other sorts of behavior.

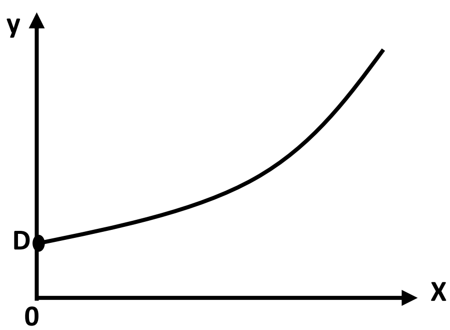
Important Notes

- ✓ Our concern here would be on those extreme points other than the end points (i.e., that represents the case where the explanatory variable takes the value of zero). This is because we mostly will consider the domain consisting of non-negative real numbers.
- ✓ The concern here is about relative (local) rather that absolute (global) extreme points.
- ✓ A function can have several relative extreme points but only one (unique) absolute extreme point.
- ✓ If the objective function is a constant function, all values of the choice variable x will result in the same value of y.



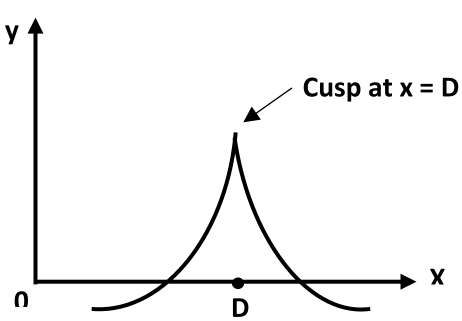
all points (a,b,c) may be considered as maximum or minimum or neither.

- ✓ If a function is strictly increasing, there will be no finite maximum if the set of nonnegative real numbers is taken to be its domain.



However, we may consider the end **point D** (the y-intercept) as an absolute/global minimum in the range of the function.

- ✓ If the first derivative $f'(x)$ does not exist (no derivative is defined), the tangent line does not exist; that is, there is a cusp on the graph at x0. The relative extremum may exist.



Non-continuous function

- ✓ The necessary condition for having an extreme (maximum/minimum) point is that the first derivative of the function at that point equals zero.
- ✓ The sufficient condition is the change in the sign of the first derivative before and after that point under consideration.
- ✓ If the 1st derivative of a function equals zero at any point, this does not necessarily imply that this is an extreme point (it may be an inflection point). That is, any extreme point is a stationary point but the opposite is not necessarily true: a stationary point may be an extreme (maximum/minimum) point or an inflection point.
- ✓ According to the first derivative, if $f'(x_0) > 0$, this means the value of the function tends to increase.
- ✓ According to the first derivative, if $f'(x_0) < 0$, this means the value of the function tends to decrease.
- ✓ According to the second derivative, if $f''(x_0) > 0$, this means the slope of the curve (function) tends to increase.
- ✓ According to the second derivative, if $f''(x_0) < 0$, this means the slope of the curve (function) tends to decrease.
- ✓ A positive first derivative with a negative second derivative (i.e., when $f'(x_0) > 0$ & $f''(x_0) < 0$) indicates that the slope of the curve is positive but decreasing (i.e., the value of the function is increasing at a decreasing rate).
- ✓ A negative first derivative with a positive second derivative (i.e., when $f'(x_0) < 0$ & $f''(x_0) > 0$) indicates that the slope of the curve is negative and increasing (for example., from -11 (the smaller) to -10 (the larger))

Questions

Question 1: Relative Extrema- First Order

Q1) By utilizing the first derivative test, find the relative extrema of the function and verify whether critical values represent relative maxima of minima. Graph your results.

$$\mathbf{y = x^3 - 12x^2 + 36x + 8}$$

Solution:

Step1

- According to **the first derivative test,** if $\frac{dy}{dx} = 0$, then we can state that the points we are looking for are critical values.

$$f'(x) = \frac{dy}{dx} = \mathbf{3x^2 - 24x + 36}$$

- Critical values are those satisfy the following condition, $\frac{dy}{dx} = 0$, thus:

$$\mathbf{3x^2 - 24x + 36 = 0}$$

By factoring the polynomial or by applying the quadratic formula, we get:

$x_1^* = 6 \rightarrow f(6) = 8$

$x_2^* = 2 \rightarrow f(2) = 40$

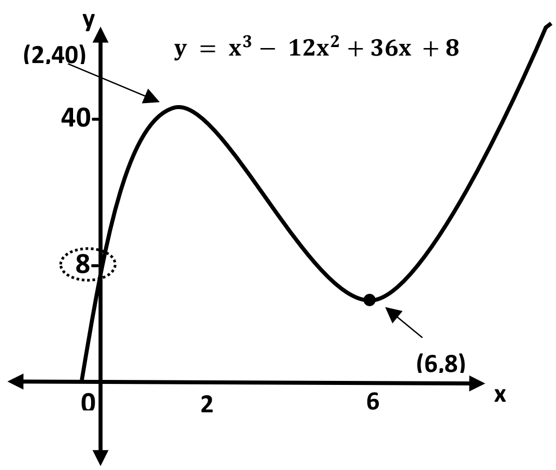
Thus, stationary points are: (6,8) & (2,40)

Step2

To apply the first-derivative test, we need to test values for $x \gtrless 6$ and $x \gtrless 2$

- For the immediate neighborhood of $x = 6$, we get $f'(x) < 0$ for $x < 6$ and $f'(x) > 0$ for $x > 6$; thus, the value of the function $f(6) = 8$ is a relative minimum.
- For the immediate neighborhood of $x = 2$, we get $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$; thus, the value of the function $f(2) = 40$ is a relative maximum.

Graphing the results:



Question 2: Relative Extrema- First Order

Q2) By utilizing the first derivative test, find the relative extrema of the following average-cost function and verify whether critical values represent relative maxima of minima. Graph your results.

$$\mathbf{AC} = \mathbf{f(Q)} = \mathbf{Q^2} - \mathbf{5Q} + \mathbf{8}$$

Solution:

Step1

- According to **the first derivative test,** if $\frac{dy}{dx} = 0$, then we can state that the points we are looking for are critical values.

$$f'(Q) = \frac{dAC}{dQ} = \mathbf{2Q - 5}$$

- Critical values are those satisfy the following condition, $\frac{dAC}{dQ} = 0$, thus:

$$\mathbf{2Q - 5 = 0}$$

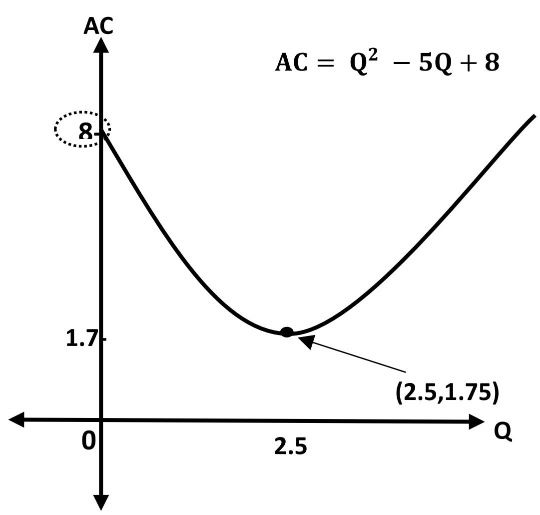
Thus, the critical value is:

$$\mathbf{Q^* = 2.5}$$

Step2

To apply the first-derivative test, we need to test values for $Q \gtrless 2.5$.

- For the immediate neighborhood of $Q = 2.5$, we get $f'(2.4) < 0$ and $f'(2.6) > 0$; thus, the value of the function $f(2.5) = 1.75$ is a relative minimum.



Question 3: The Stationary Points

Q3) Find the stationary points of the function $y = x^3 - 3x + 2$ and determine whether we have maximum or minimum points. Graph your results.

Solution:

Step1

- According to **the first derivative test,** If $\frac{dy}{dx} = 0$, then we can state that the points we are looking for are critical values.

$$\frac{dy}{dx} = 3x^2 - 3 \quad (1)$$

- Critical values are those satisfy the following condition, $\frac{dy}{dx} = 0$, thus:

$$3x^2 - 3 = 0$$

$$3(x^2 - 1) = 0 \quad \text{(factorising)}$$

$$3(x - 1)(x + 1) = 0 \quad \text{(factorising the difference of two squares)}$$

Thus, the critical values are: $x_1^* = 1$ or $x_2^* = -1$

$x_1^* = 1 \rightarrow y = f(1) = 0$ (first stationary value associated with the critical value x=1)

$x_2^* = -1 \rightarrow y = f(-1) = 4$ (second stationary value associated with the critical value x=-1)

The two stationary points are: $(1, 0)$ and $(-1, 4)$

Step2

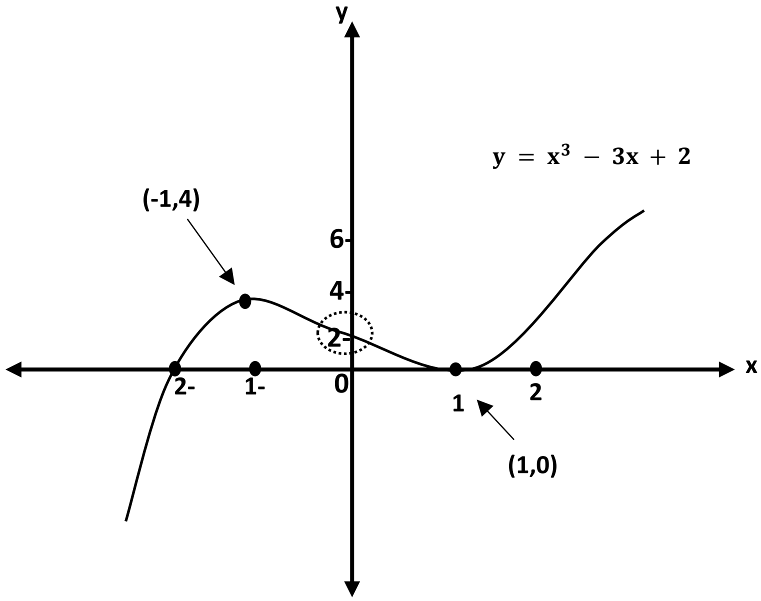
$$\frac{dy}{dx} = 3x^2 - 3$$

$$\frac{d^2y}{dx^2} = 6x$$

According to the second derivative test, we need to determine whether $\frac{d^2y}{dx^2} \gtreqless 0$,

when $x = 1 \rightarrow \frac{d^2y}{dx^2} = 6x = 6 \rightarrow$ A minimum point

when $x = -1 \rightarrow \frac{d^2y}{dx^2} = 6x = -6 \rightarrow$ A maximum point



Note: To find and graph the value of x when y = zero, you need to solve the following equation: $x^3 - 3x + 2 = 0$, which can be factored as $(x-1)\,(x^2 + x - 2)$. That is, $x - 1 = 0$ and $x^2 + x - 2 = 0$, thus, $(x-1)(x+2) = 0$ and $(x-1) = 0$. The solutions are $x = 1$ and $x = -2$

Question 4: The Necessary and Sufficient Conditions for Profit Function

Q4) Find the necessary and sufficient conditions for the profit function $\pi = \pi(Q)$, given that the revenue function is $R(Q)$ and the cost function is $C(Q)$. What is the Economic explanations?

Solution:

$$\pi(Q) = R(Q) - C(Q)$$

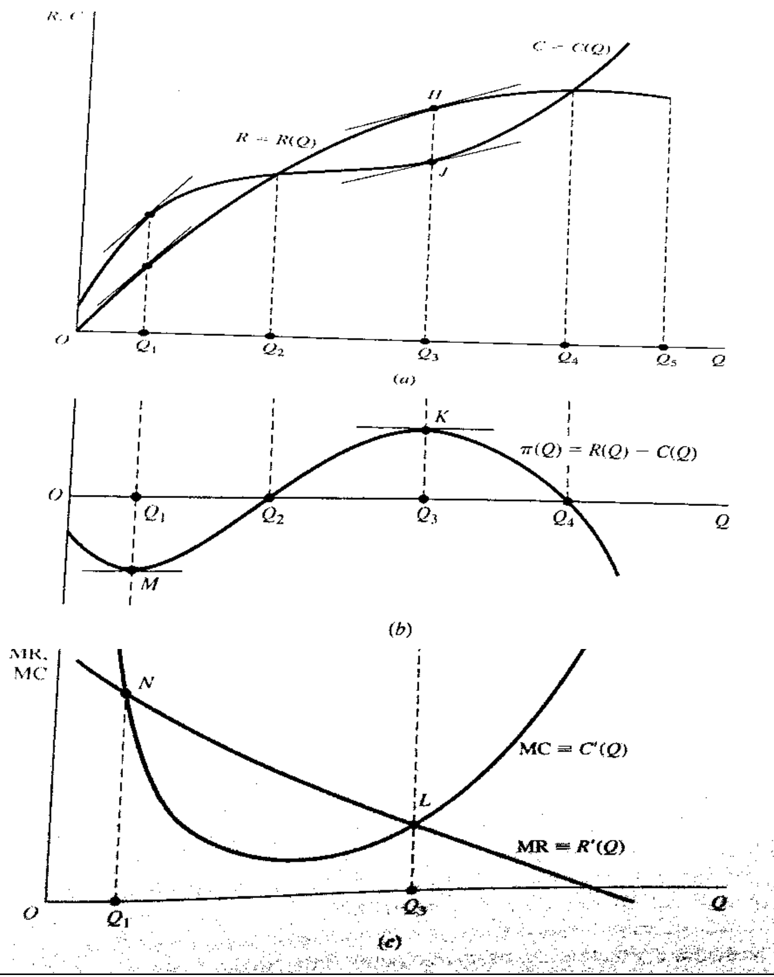
The profit-maximizing output level should satisfy the first-order necessary condition for a maximum (i.e. $\frac{d\pi}{dQ} = 0$). That is,

$\frac{d\pi}{dQ} = R'(Q) - C'(Q) = 0$ iff $R'(Q) = C'(Q)$

Thus, the optimum output Q must satisfy that MR=MC → the first-order necessary condition. For profit to be a maximum, we need to apply the second-order condition (the sufficient), that is:

$$\frac{d^2\pi}{dQ^2} = R''(Q) - C''(Q) < 0 \text{ iff } R''(Q) < C''(Q)$$

Economic explanations: If the rate of change of MR is less than the rate of change of MC (the second-order condition) at the output where MC=MR, then the output will maximize profit.



Question 5: The Necessary and Sufficient Conditions for Profit Function (Numerical)

Q5) Find the necessary and sufficient conditions for the profit function $\pi = \pi(Q)$, given that:

- **The revenue function is $R(Q) = 1200\,Q - 2Q^2$**
- **The cost function is $C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$.**

Solution:

$$\pi(Q) = R(Q) - C(Q)$$
$$\pi(Q) = -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

The critical values are obtained by the first-order derivative condition If (i.e. $\frac{d\pi}{dQ} = 0$):

$$\frac{d\pi}{dQ} = -3Q^2 + 118.5Q - 328.5 = 0$$

The critical values are: $Q_1^* = 3$ & $Q_2^* = 36.5$
Step 2
According to the second-order derivative condition:

$$\frac{d^2\pi}{dQ^2} = -6Q + 118.5$$

When $Q_1^* = 3$ → $\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \ > 0$

When $Q_2^* = 36.5$ → $\frac{d^2\pi}{dQ^2} = -6Q + 118.5 \ < 0$

Thus, the profit maximizing output is $Q_2^* = 36.5$
By substituting Q_2^* into the profit function, we can get the maximized profit Maximized profit $\pi^* = 16318.44$

Another solution:

Find $MR = 1200 - 4Q$
Find $MC = 3Q^2 - 122.5Q + 1528.5$
The first order condition: MR=MC

$$1200 - 4Q = 3Q^2 - 122.5Q + 1528.5$$

Thus, by equating the two equations, we can find the critical values of Q (i.e. $Q_1^* = 3$ & $Q_2^* = 36.5$)