Chapter 15

Differential Equations

A **differential equation** is a mathematical equation that relates some function with its derivatives. We will examine the first-order linear differential equation, which takes the following general form:

$$\frac{dy}{dt} + \mathrm{U(t)y} = \mathrm{W(t)} \quad \textbf{(the general form)}$$

Two possilbe cases of the first order linear differntial equation

Homogenous Case	**Nonhomogenous Case**
$\boldsymbol{\frac{dy}{dt} + ay = 0}$ or $\frac{1}{y}.\frac{dy}{dt}$=-a	$\boldsymbol{\frac{dy}{dt} + ay = b}$

➢ Homogenous Case of the First Order Linear Differential Equation

It occurs when U in the general form takes a constant (a) and W(t) = zero.

That is: a≠0, and b=0

It has two solutions[1]

a) A general solution: $y(t) = A\, e^{-at}$

b) A definite solution "a particular solution": $y(t) = \mathrm{y(0)}\, e^{-at}$

Example (1):

Solve the equation $\frac{dy}{dt} + 4y = 0$, with the initial condition y(0) = 1.

The solution

Here we have a homogeneous case with a = 4 and b = 0, thus,

$$y(t) = \mathrm{y(0)}\, e^{-at}$$
$$y(t) = \boldsymbol{e^{-4t}}$$

Case of: a≠0, and b=0

➢ Nonhomogeneous Case of the First Order Linear Differential Equation

It consists of **two** terms:

1. The complementary function (y_c):

$$y_C = A\, e^{\text{-at}}$$

2. The particular integral (y_P);

$$y_P = \frac{b}{\mathrm{a}}, \qquad (\mathrm{a} \neq 0)$$

It involves any particular solution of the complete equation. It has three cases:

If y = K "a Constant", accordingly $\frac{dy}{\mathrm{dt}} = 0$, since the constant doesn't change over time.

Then, $ay = b$ or $y = \frac{b}{\mathrm{a}}$,

In this case we have $y_P = \frac{b}{\mathrm{a}}$, $(\mathrm{a} \neq 0)$

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[1] Notes on the solutions of differential equations:

(a) The solution is not a numerical value, but rather a function y(t).

(b) The solution is free of any derivative or differential expressions.

Note that: the sum of the complementary function and the particular integral constitutes the general solution of the complete equation: $\frac{dy}{dt} + ay = b$

That is the general solution of the complete equation when a $\neq$ 0 is:

$$y(t) = y_{\mathrm{C}} + y_{\mathrm{P}} = A\, e^{-\mathrm{at}} + \frac{b}{\mathrm{a}}$$

[the general solution, case of a $\neq$ 0]

Case of a≠0 and b≠0

At y(0), that is by setting t=0, we find that:

$$y(0) = y_{\mathrm{C}} + y_{\mathrm{P}} = A\, e^{-\mathrm{a}(0)} + \frac{b}{\mathrm{a}} = A + \frac{b}{\mathrm{a}}$$

$$\text{Or } A = y(0) - \frac{b}{\mathrm{a}}$$

Then we can rewrite the general solution when a $\neq$ 0 as:

$$y(t) = \left[y(0) - \frac{b}{\mathrm{a}}\right] e^{-\mathrm{at}} + \frac{b}{\mathrm{a}}$$

[the definite solution, case of a $\neq$ 0]

Note that: a **general solution** (includes A) and a **definite solution** (does not include A).

Example (2):

Solve the equation $\frac{dy}{dt} + 2y = 6$, with the initial condition y(0) = 10.

<u>The solution</u>

Here we have a nonhomogeneous case with a = 2 and b = 6, thus,

$$y(t) = \left[y(0) - \frac{b}{\mathrm{a}}\right] e^{-\mathrm{at}} + \frac{b}{\mathrm{a}}$$

Then,

$$y(t) = \left[10 - \frac{6}{2}\right] e^{-2\mathrm{t}} + \frac{6}{2} = \boldsymbol{7e^{-2t} + 3}$$

Now what if a= 0, in this case, the differential equation is of the extremely simple form:

$$\frac{dy}{dt} = b$$

Its general solution will be:

$$y(t) = \mathrm{bt} + \mathrm{C}$$

Its complementary function $y_{\mathrm{C}} = A\, e^{-\mathrm{at}} = = A\, e^{-(0)\mathrm{t}} = \mathrm{A}$ (Arbitrary Constant)

Its particular integral $(y_{\mathrm{P}}) = bt,\ (\mathrm{a} = 0)$

The general solution when a = 0 will be:

$$y(t) = y_{\mathrm{C}} + y_{\mathrm{P}} = \mathrm{A} + \mathrm{bt}$$

[the general solution, case of a = 0]

a=0 and b≠0

By definitizing the arbitrary constant, we find the definite solution to be:

$$y(t) = \mathrm{y}(0) + \mathrm{bt}$$

[the definite solution, case of a = 0]

Example (3):

Solve the equation $\frac{dy}{dt} = 2$, with the initial condition y(0) = 5.

<u>The solution</u>

Here we have a nonhomogeneous case with a = 0 and b = 2, thus,

$$y(t) = \mathrm{y}(0) + \mathrm{bt}$$

Then,

$$y(t) = 5 + 2\mathrm{t}$$

The economic application: dynamics of market price

The Framework:

Suppose the demand and the supply function of a particular commodity are given by:

$$Q_d = \alpha - \beta P \quad (\alpha, \beta > 0)$$
$$Q_S = -\gamma + \delta P \quad (\gamma, \delta > 0)$$

The equilibrium price should be:

$$P^* = \frac{\alpha + \gamma}{\beta + \delta}$$

The question needed to be answered is "given the sufficient time for the adjustment process, does the time path tend to bring the price to the equilibrium level? Or does the time path P(t) tends to converge to P* as $T \rightarrow \infty$

How could you find the time path?

Let's assume that the rate of change of price (with respect to time) is directly proportional to the excess demand (Q_d-Q_s). that is,

$$\frac{dp}{dt} = j(Q_d - Q_s)\,, \quad (j > 0)$$

j: a constant adjustment coefficient.

So,

$$\frac{dp}{dt} = \alpha - \beta P + \gamma - \delta P = j(\alpha + \gamma) - j(\beta + \delta)P$$
$$\frac{dp}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

It has the form of differential equation

$$\boldsymbol{\frac{dy}{dt} + ay = b}$$

Its general solution is:

$$y(t) = y_C + y_P = A\, e^{-at} + \frac{b}{a}$$

And its definite solution is:

$$y(t) = \left[y(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a}$$

Case of a≠0 and b≠0

Accordingly,

$$P(t) = \left[P(0) - \frac{\alpha + \gamma}{\beta + \delta}\right] e^{-j(\beta+\delta)t} + \frac{\alpha + \gamma}{\beta + \delta}$$

Or

$$P(t) = \left[P(0) - P^*\right] e^{-Kt} + P^*, \quad K = j(\beta + \delta)$$

Note that:

(P^*) represents <u>the particular integral (y_P): which can be explained as the intertemporal equilibrium level.</u>

$\left(P(0) - P^*\right)$ represents the complementary function <u>y_C</u>: the deviation from the equilibrium. Thus, the dynamic stability requires the asymptotic vanishing of the complementary function.

That is,

If (K>0), as $T \rightarrow \infty$, the time path will lead the price toward the equilibrium position. That is the P(t) converges to the level P^*. The equilibrium is said to be **dynamically stable**.

The previous equilibrium can be interpreted as intertemporal equilibrium.

From the previous equation, we can observe three possible cases:

a) If **$P(0) = P^*$:** the time path of price can be drawn as horizontal straight line.
b) If **$P(0) > P^*$:** the time path of price approach the equilibrium level P^* from above.
c) If **$P(0) < P^*$:** the time path of price approach the equilibrium level P^* from below.

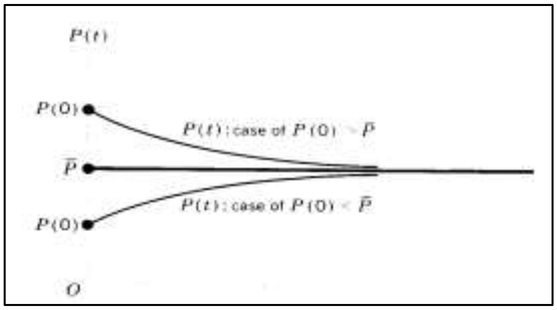
In general,

To have a dynamic stability, the deviation of the time path from equilibrium must either be identically zero (case a), or steadily decrease with time (case b and c)

Important note:

In the previous model, the particular integral (y_P) is a constant (i.e., P^*), so we have a stationary equilibrium.

If the particular integral (y_p) is non-constant, we may interpret it as a moving equilibrium rather than a stationary equilibrium.



Example,

If we have the following demand and supply functions:

$$Q_d = 10 - 2P$$
$$Q_S = -5 + 3P$$

If we assume that the rate of change of price (with respect to time) is directly proportional to the excess demand (Q_d-Q_s), as follows:

$$\frac{dp}{dt} = 0.2(Q_d - Q_s)$$

a) Find the equilibrium price.
b) Find the definite solution of the time path of the price.
c) Could you consider the dynamic equilibrium as stationary?

The Solution

a) At equilibrium,

$$Q_d = Q_S$$
$$10 - 2P = -5 + 3P$$
$$P^* = 3$$

b) To find the time path:

$$\frac{dp}{dt} = 0.2(10 - 2P + 5 - 3P)$$
$$\frac{dp}{dt} = 0.2(15 - 5P) = 3 - P$$

Or

$$\frac{dp}{dt} + P = 3$$

Differential equation (definite solution)

$$y(t) = \left[y(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a}$$

Case of a≠0 and b≠0

a=1; b=3; y(0)=P(0)

$$P(t) = \left[P(0) - \frac{b}{a}\right] e^{-at} + \frac{b}{a}$$
$$P(t) = \left[P(0) - P^*\right] e^{-at} + P^*$$
$$P(t) = [P(0) - 3] e^{-t} + 3$$

c) **Note that:**

$Y_c = [P(0) - 3]e^{-t}$

$Y_p = 3$

Since $Y_p = 3$ (a constant) → this is a case of a stationary dynamic equilibrium.

The end