Chapter 13

Further Topics in Constrained Optimization Problems

<u>The concept</u>

- The envelope theorem concerns how the optimal value for a particular function changes when a parameter of the function changes.
- In economic optimization problems, the objective functions that we try to maximize/minimize often depend on parameters, like prices. We want to find out how the optimal value is affected by changes in the parameters.

Envelope Theorem for Unconstrained Optimization (maxima)

The envelope theorem states that the change in the optimal value of a function with respect to a parameter of that function can be found by partially differentiating the objective function while holding *x* (or several *x*'s) at its optimal value.

That is,

$$\frac{dy^*}{da} = \frac{dy}{da}\ \{x = x^*(a)\}$$

To get the idea, suppose that *y* is a function of *x*

$$y = -x^2 + ax$$

Suppose we are interested in how y^* changes as *a* changes

<u>Steps:</u>

<u>Option (1)</u>:

- To calculate the slope of the function, we must solve for the optimal value of *x* for any value of *a*

$$\frac{dy}{dx} = -2x + a = 0$$

$$x^* = \frac{a}{2}$$

Substituting, we get

$$y^* = -(x^*)^2 + a(x^*) = -\left(\frac{a}{2}\right)^2 + a\left(\frac{a}{2}\right)$$

$$y^* = -\left(\frac{a^4}{4}\right) + \left(\frac{a}{2}\right)^2 = \frac{a^2}{4}$$

Therefore,

$$\frac{dy^*}{da} = \frac{2a}{4} = \frac{a}{2} = x^*$$

<u>Option (2)</u>:

But, we can save time by using the **envelope theorem**

For small changes in *a*, $\frac{dy^*}{da}$ can be computed by holding *x* at x^* and calculating $\frac{dy}{da}$ directly from *y*.

- $\frac{dy}{da} = x$
- Holding $x = x^*$
- $\frac{dy}{da} = x^* = \frac{a}{2}$

The Envelope Theorem (The Proof)

The envelope theorem says that only the direct effects of a change in an exogenous variable need to be considered, even though the exogenous variable may also enter the maximum-value function indirectly as part of the solution to the endogenous choice variables.

To get the idea, consider the following unconstrained maximization problem with two choice variables x and y and one parameter ∅.

$$Maximize\ U = f(x, y; \emptyset)$$

The F.O.C

$$f_x(x, y; \emptyset) = f_y(x, y; \emptyset) = 0$$

The implicit solutions are:

$$x^* = x^*(\emptyset);\ y^* = y^*(\emptyset)$$

If we substitute these solutions into the objective function, we obtain a new function:

$$\boldsymbol{V}(\emptyset) = f(x^*(\emptyset), y^*(\emptyset), \emptyset)$$

Note that: the previous function is the value of f when the values of x and y are those that maximize $f(x, y; \emptyset)$. Therefore, $\boldsymbol{V}(\emptyset)$ is the maximum-value function (or indirect objective function.

If we differentiate V with respect to $\emptyset$, we get

$$\frac{\boldsymbol{dV}}{\boldsymbol{d\emptyset}} = f_x \frac{\delta x^*}{\delta\emptyset} + f_y \frac{\delta y^*}{\delta\emptyset} + f_\emptyset$$

However, from the first condition we know that $f_x = f_y = 0$. Therefore, the first two terms disappear and the result becomes: $\frac{\boldsymbol{d}\mathrm{V}}{\boldsymbol{d\emptyset}} = f_\emptyset$.

This result says that, at the optimum, as $\emptyset$ varies, with x^* and y^* allowed to adjust, the derivative $\frac{\boldsymbol{dV}}{\boldsymbol{d\emptyset}}$ **gives the same result as if** x^* and y^* are treated as constants.

Note (1): the $\emptyset$ enters the maximum-value function in three places: one direct and two indirect through x^* and y^*.

Note (2): the previous equation $\frac{\boldsymbol{d}\mathrm{V}}{\boldsymbol{d\emptyset}} = f_\emptyset$ shows that, at the optimum, only the direct effect of $\emptyset$ on the objective function matters. (the previous conclusion is the essence of the envelope theorem).

Envelope theorem for constrained maxima

If we have an objective function (U), two choice variables (x, y) and one parameter ($\emptyset$) and the following constraint: $g(x, y; \emptyset) = 0$,

The problem become:

$$Maximize\ U = f(x, y; \emptyset)$$

s.t

$$g(x, y; \emptyset) = 0$$

The Lagrangian for the above-optimization problem is:

$$Z = f(x, y; \emptyset) + \lambda[0 - g(x, y; \emptyset)]$$

F.O.C

$$Z_x = f_x - \lambda g_x = 0$$
$$Z_y = f_y - \lambda g_y = 0$$
$$Z_\lambda = -g(x, y; \emptyset) = 0$$

Solving the previous system of equations gives us:

$$x = x^*(\emptyset);\ y = y^*(\emptyset);\ \lambda = \lambda^*(\emptyset)$$

How does $\boldsymbol{V}(\emptyset)$ **change as** $\emptyset$ **changes?**

By substituting the optimal values into the Lagrangian yields the indirect objective function, **maximum-value function** $\boldsymbol{V}(\emptyset)$, which is the maximum value of y for any $\emptyset$ and x that satisfy the constraint.

$$\boldsymbol{V}(\emptyset) = f(x^*(\emptyset), y^*(\emptyset), \emptyset) + \mathrm{g}[(x^*(\emptyset), y^*(\emptyset), \emptyset)]$$

By differentiating V with respect to $\emptyset$. This yields:

$$\boldsymbol{V}(\emptyset) = f_x \frac{\delta x^*}{\delta\emptyset} + f_y \frac{\delta y^*}{\delta\emptyset} + f_\emptyset + g_x \frac{\delta x^*}{\delta\emptyset} + g_y \frac{\delta y^*}{\delta\emptyset} + g_\emptyset = 0$$

$$\frac{d\boldsymbol{V}}{d\emptyset} = (f_x - \lambda^* g_x) \frac{\delta x^*}{\delta\emptyset} + [f_y - \lambda^* g_y] \frac{\delta y^*}{\delta\emptyset} + f_\emptyset - \lambda^* g_\emptyset = Z(\emptyset)$$

Where $Z_\emptyset$ is the partial derivative of the Lagrangian function with respect to $\emptyset$, holding all other variables constant.

The Interpretation of Lagrange Multiplier (Envelope Theorem).
In the consumer choice problem, we derive the result that the lagrange multiplier λ represent the change in the value of the lagrange function when the consumer's budget changed. That is, the λ can be interpreted as the **<u>marginal utility of income</u>**.

Now let us derive a more general interpretation of the lagrange multiplier with the assistance of the envelope theorem. **Consider the following problem**:

$$\boldsymbol{Maximize}\ U = f(x,y)$$
$$\boldsymbol{Subject}\ to\ g(x,y) = C$$

The Lagrangian for this problem is:

$$Z = f(x,y) + \lambda[C - g(x,y)]$$

F.O.C

$$Z_x = f_x(x,y) - \lambda g_x(x,y) = 0 \quad (1)$$
$$Z_y = f_y(x,y) - \lambda g_y(x,y) = 0 \quad (2)$$
$$Z_\lambda = C - g(x,y) = 0 \quad (3)$$

From (1) and (2), we get:

$$\lambda = \frac{f_x}{g_x} = \frac{f_y}{g_y}$$

The previous equation gives us the the condition that the slope of the level curve (indifference curve) of the objective function must equal the slope of the constraint at the optimum (i.e. $\frac{f_x}{f_y} = \frac{g_x}{g_y}$).

The F.O.C solutions:

$$x^* = x^*(C)$$
$$y^* = y^*(C)$$
$$\lambda^* = \lambda^*(C)$$

Substituting the F.O.C solutions into the Lagrangian yields the maximum-value function,

$$Z^*(C) = f(x^*(C), y^*(C)) + \lambda^*(C)[C - g(x^*(C), y^*(C))]$$

Differentiating with respect to **C** yields:

$$\frac{dZ^*}{dC} = f_x\frac{\delta x^*}{\delta C} + f_y\frac{\delta y^*}{\delta C} + [C - g(x^*(C), y^*(C))]\frac{\delta \lambda^*}{\delta C} - \lambda^*(C)g_x\frac{\delta x^*}{\delta C} - \lambda^*(C)g_y\frac{\delta x^*}{\delta C} + \lambda^*(C)\frac{dC}{dC}$$

By rearranging we get:

$$\frac{dZ^*}{dC} = [f_x - \lambda^* g_x]\frac{\delta x^*}{\delta C} + [f_y - \lambda^* g_y]\frac{\delta y^*}{\delta C} + [C - g(x^*), y^*(C))]\frac{\delta \lambda^*}{\delta C} + \lambda^*(C)$$

From (1) & (2) & (3), the three terms in brackets are all equal to zero. Therefore this expression simplifies to:

$$\frac{dZ^*}{dC} = \lambda^*(C)$$

The Interpretation:
It shows that the optimal value λ^* measures the rate of change of the maximum value of the objective function when C changes, and is for this reason referred to as the "shadow price" of C.

Demand Functions

An indirect utility function specifies the maximum utility that can be obtained given prices, income, and the utility function.

An expenditure function specifies the minimum expenditure required to obtain a fixed level of utility given the utility function and the prices of consumption goods.

- ### The Marshallian Demand: The Consumer's Ordinary Demand Functions (the Primal Problem)

Q1) If we have the following primal problem:

$$Maximize\ U = U(x, y)$$
$$S.t.\ P_x x + P_y Y = B$$

Where x and y are consumption goods, and B is the consumer's income. P_x and P_y are market prices.

a) What will be the optimal levels of goods x (x^m) and good y (y^m)?

The Lagrangian function for the previous equation are:

$$Z = U(x, y) + \lambda\left(B - P_x x - P_y Y\right)$$

The F.O.C

$$Z_x = U_x - \lambda P_x = 0 \ \ (1)$$
$$Z_y = U_y - \lambda P_y = 0 \ \ (2)$$
$$Z_\lambda = B - P_x x - P_y Y = 0 \ \ (3)$$

From the above system of equations, we obtain a solution for x^m & y^m , & λ^m as a function of the exogenous variables B, P_x , and P_y. that is,

$$x^m = x^m\left(P_x, P_y, B\right)$$
$$y^m = y^m\left(P_x, P_y, B\right)$$
$$\lambda^m = \lambda^m\left(P_x, P_y, B\right)$$

They are commonly referred to as "**Marshallian**" demand functions.

Substituting the solutions x^m and y^m into the **utility** function yields:

$$U^* = U^*\left(x^m\left(P_x, P_y, B\right), y^m\left(P_x, P_y, B\right)\right) \equiv V\left(P_x, P_y, B\right)$$

Where V is the indirect utility function- a maximum-value function, showing the maximum attainable utility.

- ### The Hicksian Demand: The Consumer's Compensated Demand Functions (Dual Problem)

Q2) Now let's consider a related **dual problem** for the consumer with the objective of minimizing the expenditure on x and y while maintaining a fixed utility level (U) derived from the previous primal problem in the (Q1).

That is, the dual problem will be:

$$\text{Minimize E} = P_x x + P_y Y$$
$$Subject\ to\ \ U(x, y) = U^*$$

The Lagrangian function will be:

$$Z^d = P_x x + P_y Y + \mu[U^* - U(x, y)]$$

The F.O.C

$$Z_x^d = P_x - \mu U_x = 0 \ \ (1)$$
$$Z_y^d = P_y - \mu U_y = 0 \ \ (2)$$
$$Z_\mu^d = U^* - U(x, y) = 0 \ \ (3)$$

From the above system of equations, we obtain a solution for x^h & y^h , & μ^h as a function of the exogenous variables U^*, P_x , and P_y. that is,

$$x^h = x^h\left(P_x, P_y, U^*\right)$$
$$y^h = y^h\left(P_x, P_y, U^*\right)$$
$$\mu^h = \mu^h\left(P_x, P_y, U^*\right)$$

Where x^h and y^h are the compensated demand functions, in which real income held constant. They are commonly referred to as "**Hicksian**" demand functions.

Substituting the solutions x^h and y^h into the **expenditure** function yields:

$$E^*\left(P_x, P_y, U^*\right) = P_x x^h\left(P_x, P_y, U^*\right) + P_y y^h\left(P_x, P_y, U^*\right)$$

Where E^* is the Expenditure function- a minimum-value function, showing the minimum expenditure needed to attain the utility level U^*.

Important Notes (and Properties) about the Dual Problem of "Marshallian" and "Hicksian"

(1) From equations (1) and (2) of the F.O.C of the marshallian demand functions:

$$Z_x = U_x - \lambda P_x = 0 \quad (1)$$
$$Z_y = U_y - \lambda P_y = 0 \quad (2)$$

We can conclude: $\frac{P_x}{P_y} = \frac{U_x}{U_y}$

The previous condition is the tangency condition in which the consumer choose the optimal bundle where the slope of the indifference curve equals the slope of the budget constraint.

Note: The tangency condition is identical for both problems ("Marshallian" and **"Hicksian")**

(2) When the target level of utility in the minimization problem is set equal to the value (U^*) obtained from the maximization problem, we find that the solutions to both the maximization problem and the minimization problem produce identical values for x and y. that is,

$$x^m\left(P_x, P_y, B\right) = x^h\left(P_x, P_y, U^*\right)$$
$$y^m\left(P_x, P_y, B\right) = y^h\left(P_x, P_y, U^*\right)$$

However, the solutions are functions of different exogenous variables. That is, the comparative static analysis will produce different results.

(3) The Lagrangian multipliers (λ and μ) are reciprocal to each other. Since from the Marshallian demand function, $\lambda = \frac{U_x}{P_x}$, while from the Hicksian demand function, $\mu = \frac{P_x}{U_x}$, thus, , $\lambda = \frac{1}{\mu}$ or $\lambda^m = \frac{1}{\mu^h}$

Other Simple Property of Marshallian Demand Functions

- If we were to double all prices and income, the optimal quantities demanded will not change

– Notice that the budget constraint does not change (the slope does not change, the crossing with the axis do not change either)

Changes in Income

- Since p_x/p_y does not change, the *MRS* will stay constant
- An increase in income will cause the budget constraint out in a parallel fashion (MRS stays constant)

What is a Normal Good?

- A good x_i for which $\partial x_i/\partial I \geq 0$ over some range of income is a <u>normal</u> good in that range

What is an inferior Good?

- A good x_i for which $\partial x_i/\partial I < 0$ over some range of income is an <u>inferior</u> good in that range

Changes in a Good's Price

- A change in the price of a good alters the slope of the budget constraint (p_x/p_y)
  Consequently, it changes the *MRS* at the consumer's utility-maximizing choices
- When a price changes, we can decompose consumer's reaction in two effects:
  - substitution effect
  - income effect
- Even if the individual remained on the same indifference curve when the price changes, his optimal choice will change because the *MRS* must equal the new price ratio
  (The substitution effect)
- The price change alters the individual's real income and therefore he must move to a new indifference curve
  (The income effect)

Examples (Demand Functions)

Example (1) Consider a consumer with the following utility function:

$$U = xy$$

If you know that the consumer faces a budget constraint of B and is given prices are p_x and p_y.

a) Find the consumer's Marshallian demand functions.
b) Check the second order condition
c) Derive the indirect utility function
d) Derive the dual problem

Solution:

The choice problem is:

$$Maximize\ U = xy$$
$$Subject\ to\ \ p_x x + p_y y = B$$

The Lagrangian for the problem is:

$$Z = xy + \lambda\left[B - p_x x - p_y y\right]$$

The F.O.Cs

$$Z_x = y - \lambda p_x = 0$$
$$Z_y = x - \lambda p_y = 0$$
$$Z_\lambda = B - p_x x - p_y y = 0$$

Solving the system of equations, we obtain:

$$x^m = \frac{B}{2p_x}$$
$$y^m = \frac{B}{2p_y}$$
$$\lambda^m = \frac{B}{2p_x p_y}$$

(the x^m and y^m are the Marshallian demand functions)

The S.O.Cs

$$|\bar{H}| = \begin{vmatrix} 0 & 1 & -p_x \\ 1 & 0 & -p_y \\ -p_x & -p_y & 0 \end{vmatrix} = 2p_x p_y > 0$$

The solution represents a maximum.

➢ The indirect utility function

We can now derive the indirect utility function $V(p_x, p_y, B)$ for the above problem by substituting x^m and y^m into the utility function:

Since $U = xy$

$$V(p_x, p_y, B) = \left(\frac{B}{2p_x}\right)\left(\frac{B}{2p_y}\right) = \frac{B^2}{4p_x p_y}$$

- The dual problem

  Since $U^* = \frac{B^2}{4p_x p_y}$, by rearranging the terms, we can get B

$$B = \left(4p_x p_y U^*\right)^{1/2} = 2\, p_x^{\ 1/2} p_y^{\ 1/2} U^{*1/2}$$

  The dual problem should represent the minimum-expenditure function, in which the function E should be equal to the given budget amount B of the primal problem. Therefore, we can immediately conclude from the preceding equation that:

$$E\left(p_{x,} p_{y,} U^*\right) = B = 2\, p_x^{\ 1/2} p_y^{\ 1/2} U *^{1/2}$$

Example (2) Consider the dual problem of the cost minimization given a fixed level of utilty related to example (1). Letting U* denote the target level of utility and the expenditure function is given by (E), that is,

$$E = p_x x + p_y y$$
$$s.t.\ xy = U^*$$

a) Find the consumer's compensated (Hicksan) demand functions.
b) Check the second order condition
c) Derive the minimum-value function
d) Derive the dual problem

Solution:

The Lagrangian for the problem will be

$$Z^d = p_x x + p_y y + \mu(U^* - xy)$$

- **The F.O.Cs:**

$$Z_x^d = p_x - \mu y = 0$$
$$Z_y^d = p_y - \mu x = 0$$
$$Z_\mu^d = U^* - xy = 0$$

  Solving the system of equations, we obtain:

$$x^h = \left(\frac{p_y U^*}{p_x}\right)^{1/2}$$
$$y^h = \left(\frac{p_x U^*}{p_y}\right)^{1/2}$$
$$\mu^h = \left(\frac{p_x p_y}{U^*}\right)^{1/2}$$

  Where x^h and y^h are the consumer's compensated (Hicksan) demand functions.

- **The S.O.Cs:**

$$|\bar{H}| = \begin{vmatrix} 0 & -\mu & -y \\ -\mu & 0 & -x \\ -y & -x & 0 \end{vmatrix} = -2xy\mu < 0$$

  It is a minimum

- Substituting x^h and y^h into the original expenditure function, we can obtain the minimum-value function (or the expenditure function), that is,

  Since $E = p_x x + p_y y$

$$E = p_x x^h + p_y y^h = p_x \left(\frac{p_y U^*}{p_x}\right)^{1/2} + p_y \left(\frac{p_x U^*}{p_y}\right)^{1/2}$$

$$E = \left(P_x\, P_y\, U^*\right)^{1/2} + \left(P_x\, P_y\, U^*\right)^{1/2} = 2P_x^{\ 1/2} P_y^{\ 1/2} U^{*1/2}$$

Important notes on the previous examples:

It should be noted that there exists two different alternative expressions for the Bordered Hessian:

The first expression is:

$$|\overline{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{vmatrix}$$

The second expression is:

$$|\overline{H}| = \begin{vmatrix} Z_{11} & Z_{12} & g_1 \\ Z_{21} & Z_{22} & g_2 \\ g_1 & g_2 & 0 \end{vmatrix}$$

In the previous example, we have used the second expression