Chapter 12

Constrained Optimization Problems

Finding the Extreme Points of a Function of Two Choice Variables with a Constraint

In the last chapter, we assumed that all the choice variables are independent of one another, in the sense that the decision made regarding one variable does not impinge upon the choices of the remaining variables.

> For instance, a two-product firm can choose any value for Q_1 and any value for Q_2 it wishes, without the two choices limiting each other.

However,

The existence of a restriction (e.g., production quota) will result in a loss of the independence between the choice variables. In such case, the firm's profit-maximizing output (Q_1 & Q_2) will be not only simultaneous but also dependent.

If a firm faces the following production quota: $Q_1 + Q_2 = 950$, it means that Q_1 & Q_2 are dependent, since the higher Q_1 is, the lower Q_2 must correspondingly be, in order to stay within the combined quota of 950. In such case, the new optimal output is called a " *Constrained Optimum*" as opposed to the "*Free Optimum*" discussed in the previous chapters.

Effects of a Constraint

The primary purpose of imposing a constraint is to consider certain limiting factors present in the optimization problem under discussion.

For example,

(1) the limitation on output choices that result from a production quota, i.e. $Q_1 + Q_2 = 950$.

(2) a consumer with the simple utility function: $U = x_1 x_2 + 2x_1$ .Since the marginal utilities are positive for all positive levels Of X_1 and X_2, to have U maximized without any constraint, the consumer should purchase an infinite amount of both goods, a solution that obviously has little practical relevance. To render the optimization problem meaningful, the purchasing power of the consumer must also be taken into account; i.e., a budget constraint should be incorporated into the problem.

Question:

What is the budget constraint and the optimization problem of the following problem?

If the consumer intends to spend a given sum, say, \$60, on the two goods and if the current prices are $P_{10} = 4$ and $P_{20} = 2$. The consumer's utility function is $U = x_1x_2 + 2x_1$.

Answer: The budget constraint can be expressed by the linear equation $4x_1 + 2x_2 = 60$

Note_1: X_1 and .X_2 are mutually dependent.

The problem now is to maximize $U = x_1x_2 + 2x_1$ subject to the constraint $4x_1 + 2x_2 = 60$

Note_2: the constraint narrow the domain and hence the range of the objective function. Before imposing the budget constraint, the domain was represented by the nonnegative values of x1 and x2. However, after imposing the budget constraint the domain is immediately reduced to the set of points lying on the budget line.

Finding the Stationary Values

The constrained maximum can easily be found by combining the constraint with the objective function by the substitution rule.

For example, find the constrained maximum objective function for the following utility function: $U = x_1x_2 + 2x_1$, given that $4x_1 + 2x_2 = 60$,

Solution:

$$X_2 = 30 - 2x_1,$$

$$U = x_1(30 - 2x_1) + 2x_1 = 32x_1 - 2x_1^2$$

The Lagrange-Multiplier Method (λ- Lambda)

What is The Lagrange-Multiplier Method?

The method of lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to equality constraints (i.e., subject to the condition that one or more equations have to be satisfied exactly by the chosen values of the variables.

Our purpose here is to find a solution to the following problem:

$$\text{Max/Min of } F(x, y, z)$$

Subject to:

$$g(x, y, z) = K$$ **(constraint)**

<u>Solution's Steps</u>

1- Form a new function that depends on x, y, z, λ (i.e., $f\ (x, y, z, \lambda)$)

$$= f\ (x,y,z) - \lambda(g(x,y,z) - K)$$

2- Solve: (**F.O.C**)

$$fx = 0$$
$$fy = 0$$
$$fz = 0$$
$$f\lambda = 0$$

3- Step 3: substituting solutions into $f\ (x, y, z)$

4- **Step 4:** Testing the relative minimum/maximum by utilizing 2nd order condition (**S.O.C**) (i.e., **Bordered Hessian Determinant)**.

Bordered Hessian Determinant

Remember that: **Hessian** is a square matrix of second-order partial derivatives.

A **bordered Hessian** is used for the second-derivative test in certain constrained optimization problems.

The 2-Variables Case

Given the function *f* considered previously, but adding a constraint function *g* such that *g*(**x**) = C, the bordered Hessian is the Hessian of the Lagrange function:

$$Z = f(x,y) + \lambda[C - g(x,y)]$$

$$|\overline{H}| = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & \frac{\delta^2 Z}{\delta x^2} & \frac{\delta^2 Z}{\delta x \delta y} \\ g_2 & \frac{\delta^2 Z}{\delta y \delta x} & \frac{\delta^2 Z}{\delta y^2} \end{bmatrix} < 0\ "positivie\ definite"\ Minimum$$

$$|\overline{H}| = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & \frac{\delta^2 Z}{\delta x^2} & \frac{\delta^2 Z}{\delta x \delta y} \\ g_2 & \frac{\delta^2 Z}{\delta y \delta x} & \frac{\delta^2 Z}{\delta y^2} \end{bmatrix} > 0\ "Negative\ definite"\ maximum$$

The 3-Variables Case

When the objective function takes the form

$Z = f(x_1, x_2, x_3)$ subject to $g(x_1, x_2, x_3) = C$

The bordered Hessian $|\bar{H}|$ is the Hessian of the Lagrange function:

$$Z = f(x_1, x_2, x_3) + \lambda[C - g(x_1, x_2, x_3]$$

$$|\bar{H}| = \begin{bmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & Z_{11} & Z_{12} & Z_{13} \\ g_2 & Z_{21} & Z_{22} & Z_{23} \\ g_3 & Z_{31} & Z_{32} & Z_{33} \end{bmatrix}$$

Where

$$|\bar{H}_2| = \begin{bmatrix} 0 & g_1 & g_2 \\ g_1 & Z_{11} & Z_{12} \\ g_2 & Z_{21} & Z_{22} \end{bmatrix}$$

And

$$|\bar{H}_3| = \begin{bmatrix} 0 & g_1 & g_2 & g_3 \\ g_1 & Z_{11} & Z_{12} & Z_{13} \\ g_2 & Z_{21} & Z_{22} & Z_{23} \\ g_3 & Z_{31} & Z_{32} & Z_{33} \end{bmatrix}$$

Determinantal test for relative constrained extremum:

Condition	Maximum (Negative definite)	Minimum (Positive definite)
First-order necessary condition	$Z_\lambda = Z_1 = Z_2 = Z_3 = 0$	$Z_\lambda = Z_1 = Z_2 = Z_3 = 0$
Second-order sufficient	$|\bar{H}_2| > 0;\ |\bar{H}_3| < 0$	$|\bar{H}_2|;\ |\bar{H}_3| < 0$
	all the bordered principal minors must alternate in sign	all the bordered principal minors must be negative

Note: the second-order is applicable only after the first-order necessary condition has been satisfied

The n-Variables Case

When the objective function takes the form

$Z = f(x_1, x_2, \ldots.., x_n)$ subject to $g(x_1, x_2, \ldots.., x_n) = C$

The bordered Hessian $|\bar{H}|$ is the Hessian of the Lagrange function:

$$Z = f(x_1, x_2, \ldots.., x_n) + \lambda[C - g(x_1, x_2, \ldots.., x_n)]$$

$$|\bar{H}| = \begin{bmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & Z_{11} & Z_{12} & \cdots & Z_{1n} \\ g_2 & Z_{21} & Z_{22} & \cdots & Z_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ g_n & Z_{n1} & Z_{n2} & \cdots & Z_{nn} \end{bmatrix}$$

Example:

Suppose a company "A" produces two goods (e.g., x and y) at two factories (e.g., F_1 & F_2). Assume that the operating cost is given by the following function:

$$C(x,y) = 6x^2 + 12y^2$$

If the company has to produce 90 million units a year of the two goods, what will be the quantity that should be produced at each factory in order to minimize its cost?

Solution:

The main purpose of company "A" is to minimize its cost given the output constraint. That is, the objective function will be:

$$Min: C(x,y) = 6x^2 + 12y^2$$

Subject to the following constraint:

$$x + y = 90$$

Step1: By utilizing the Lagrange multiplier method (the lagrangian function):

$$f(x,y,\lambda) = 6x^2 + 12y^2 - \lambda(x + y - 90)$$

$$= 6x^2 + 12y^2 - \lambda x - \lambda y + 90\lambda$$

Step2: The partial derivatives $(fx, fy, f\lambda)$: (**F.O.C**)

$$fx = 12x - \lambda = 0 \rightarrow (1)$$
$$fy = 24y - \lambda = 0 \rightarrow (2)$$
$$f\lambda = -x - y + 90 = 0 \rightarrow (3)$$

From (1): $x = \frac{\lambda}{12}$

From (2): $y = \frac{\lambda}{24}$

Substituting the values of x & y in (3), we get:

$$= -\frac{\lambda}{12} - \frac{\lambda}{24} + 90 = 0,$$

$$we\ get\ \lambda = 720$$

substituting the value of λ in the value of x and y, we get:

$$x = \frac{\lambda}{12} = \frac{720}{12} = 60$$

$$y = \frac{\lambda}{24} = \frac{720}{24} = 30$$

Step 3: Substituting the values of x & y in the original function $C(x, y) = 6x^2 + 12y^2$, we get:

$$C(60,30) = 6(60)^2 + 12(30)^2 = 32400$$

Step 4: Testing the relative minimum/maximum by utilizing 2nd order condition (i.e., **Bordered Hessian Determinant)**.

Second-Order Conditions (S.O.C): Bordered Hessian Determinant

Example2:

Find the extremum of the following function:

$$Z = xy \textbf{ subject to } x + y = 6$$

Step1: By utilizing the Lagrange multiplier method (the lagrangian function):

$$f(x, y, \lambda) = xy + \lambda(6 - x - y)$$

Step2: The partial derivatives ($fx, fy, f\lambda$) : (**F.O.C**)

$$fx = y - \lambda = 0 \rightarrow (1)$$
$$fy = x - \lambda = 0 \rightarrow (2)$$
$$f\lambda = 6 - x - y = 0 \rightarrow (3)$$

Step 3: by utilizing the cramer's rule (or substituting method), we find that:

$$\begin{array}{cccl} & x & +y & = 6 \rightarrow (1) \\ -\lambda & & +y & = 0 \rightarrow (2) \\ -\lambda & +x & & = 0 \rightarrow (3) \end{array}$$

$\lambda = \frac{\begin{vmatrix}6 & 1 & 1\\0 & 0 & 1\\0 & 1 & 0\end{vmatrix}}{\begin{vmatrix}0 & 1 & 1\\-1 & 0 & 1\\-1 & 1 & 0\end{vmatrix}}$	$x = \frac{\begin{vmatrix}0 & 6 & 1\\-1 & 0 & 1\\-1 & 0 & 0\end{vmatrix}}{\begin{vmatrix}0 & 1 & 1\\-1 & 0 & 1\\-1 & 1 & 0\end{vmatrix}}$	$y = \frac{\begin{vmatrix}0 & 1 & 6\\-1 & 0 & 0\\-1 & 1 & 0\end{vmatrix}}{\begin{vmatrix}0 & 1 & 1\\-1 & 0 & 1\\-1 & 1 & 0\end{vmatrix}}$

Then, $\lambda^* = 3, x^* = 3, y^* = 3,$

The stationary value is $Z^* = 9$

Step 4: testing maximum or minimum:

Second order partial derivatives are:

$$fxx = 0$$

$$fxy = 1$$

$$fyy = 0$$

The bordered Hessian= $|\overline{H}| = \begin{bmatrix}0 & g_1 & g_2\\ g_1 & \frac{\delta^2 Z}{\delta x^2} & \frac{\delta^2 Z}{\delta x \delta y}\\ g_2 & \frac{\delta^2 Z}{\delta y \delta x} & \frac{\delta^2 Z}{\delta y^2}\end{bmatrix}$

The border elements are: $g_z = 1$, and $g_y = 1$

The bordered Hessian= $|\overline{H}| = \begin{bmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{bmatrix}$=2>0 (maximum)

An Interpretation (Meaning) of the Lagrange Multiplier

By performing a comparative-static analysis on the first-order condition. A change in c “the only exogenous variable on the constraint equation, would cause a shift of the constraint curve and thereby alter the optimal solution. In particular, the effect of an increase in c (a larger budget, or a larger production quota) would indicate how the optimal solution is affected by a relaxation of the constraint.

The value of the Lagrange multiplier at the solution of the problem is equal to the rate of change in the maximal value of the objective function as the constraint is relaxed. For example, in a utility maximization problem the optimal value of the Lagrange multiplier measures the marginal utility of income: the rate of increase in maximized utility as income is increased.

To sum up, a Lagrange multiplier refers to the rate of change of the value of extrema with respect to the change of the constraint constant (i.e., “C”).

Economic applications of constrained optimization problems

Maximization of a function with a constraint is common in economic situations. For example: the problem in consumer theory of maximization of the utility function with a fixed amount of wealth (income) to spend on the commodities.

Utility Maximization Under Income Constraint

After adding the constraint, the domain of the utility function is immediately reduced to the set of points lying on the budget line. This will automatically affect the range of the objective function, too; only that subset of the utility surface lying directly above the budget-constraint line will now be relevant.

Let’s assume the consumer choose only two goods, both of which have continuous, positive, marginal utility functions. The prices of goods are market determined, hence exogenous.

Given the consumer's income, B, and prices, P_x and P_y, the consumer's problem is to choose the affordable bundle that maximizes his utility.

The feasible set (budget set): it is defined as the total expenditure that cannot exceed the income, so we have:

$$xP_x + yP_y = B$$

the problem for this consumer will be:

$$\boldsymbol{U = U(x, y)\left(U_x, U_y > 0\right)}$$

St.

$$xP_x + yP_y = B$$

F.O.C

The lagrangian function will be:

$$\boldsymbol{Z} = \boldsymbol{U}(\boldsymbol{x},\boldsymbol{y}) + \lambda(\boldsymbol{B} - xP_x - yP_y)$$

F.O.C

$$\boldsymbol{Z}_\lambda = \boldsymbol{B} - xP_x - yP_y = 0$$

$$\boldsymbol{Z}_x = \boldsymbol{U_x} - \lambda P_x = 0$$

$$\boldsymbol{Z}_y = \boldsymbol{U_y} - \lambda P_y = 0$$

Since the last two equations are equivalent to

$$\frac{\boldsymbol{U_x}}{P_x} = \frac{\boldsymbol{U_y}}{P_y} = \lambda \rightarrow (1)$$

(The above condition states that the consumers must allocate their budget so as to equalize the ratio of marginal utility to price (marginal utility per dollar) for every commodity). That is, in equilibrium (optimum), these ratios should have the common value λ^*.

Note that: λ^* measures the comparative-static effect of the constraint constant on the optimal value of the objective function (i.e., $\lambda^* = \frac{\partial U^*}{\partial B}$). That is, λ^* can be interpreted as the marginal utility of money when the consumer's utility is maximized.

By restating condition (1) in the form:

$$\frac{\boldsymbol{U_x}}{\boldsymbol{U_y}} = \frac{P_x}{P_y}$$

The Indifference Curve:

The indifference curve is the combination of x and y that will yield a constant level of U. that is,

$$dU = U_x\,dx + U_y\,dy = 0$$

That means: $\frac{\partial y}{\partial x} = -\frac{\boldsymbol{U_x}}{\boldsymbol{U_y}}$ (the slope, $\frac{\partial y}{\partial x}$, must be equal to the negative of the marginal-utility ratio $\frac{\boldsymbol{U_x}}{\boldsymbol{U_y}}$ (which is the MRS).

$\frac{P_x}{P_y}$ represents the slope of the budget constraint (the budget line).

Since the budget constraint is $xP_x + \ yP_y = \ B$

It can be rewritten as:

$$y = \frac{B}{P_y} - \frac{P_x}{P_y}x$$

Accordingly, the budget line is drawn as a straight line with slope $-\frac{P_x}{P_y}$ and vertical intercept $\frac{B}{P_y}$.

Thus, to maximize utility, a consumer must allocate the budget such that the slope of the budget line ($-\frac{P_x}{P_y}$) is equal to the slope of the indifference curve $\frac{\boldsymbol{U_x}}{\boldsymbol{U_y}}$.

S.O.C

The bordered Hessian matrix will be positive if:

$$|\overline{H}| = \begin{bmatrix} 0 & P_x & P_y \\ P_x & U_{xx} & U_{xy} \\ P_y & U_{yx} & U_{yy} \end{bmatrix} = 2P_xP_yU_{xy} - P_y^2U_{xx} - P_x^2U_{yy} > 0$$

Derivation of the Demand Function

The solution to utility maximization problem gives the consumer's choice of x and y, as a function of prices and income, which we denote by x*(p_x; p_y; M) and y*(p_x; p_y; M). These are known as the generalized demand functions. Let's have the following example to get the idea:

Example: Assume that consumers utility function is of Cobb-Douglass form:

$$U\ (x, y) \ = \ x^{\alpha}\ y^{\beta} \quad (1)$$

s.t

$$xP_x + \ yP_y \leq \ M \quad (2)$$

To solve the consumers' optimization problem it is necessary to maximize (1) subject to the budget constraint (2). To solve the problem Lagrange Theorem will be used to rewrite the constrained optimization problem into a non-constrained form:

Max $Z\ (x, y, \lambda) \ = \ x^{\alpha}\ y^{\beta} + \lambda(m \ - \ P_x x - \ P_y y)$

The first order (necessary) conditions will result in:

$$P_x x + P_y y = M \rightarrow (3)$$

$$\alpha x^{\alpha-1} y^{\beta} = \lambda P_x \rightarrow (4)$$

$$\beta x^{\alpha} y^{\beta-1} = \lambda P_y \rightarrow (5)$$

Thus,

Combining (4) & (5) will result in:

$$\alpha P_y y = \beta P_x x$$

Substituting in (3), we obtain:

$$\alpha(m - P_x x) = \beta P_x x$$

Thus,

$$\boldsymbol{x} = \frac{\alpha}{\alpha+\beta} \cdot \frac{\boldsymbol{m}}{P_x} \quad \textbf{(6)} \rightarrow \text{(This is the demand function for the x)}$$

Note: When the price of the good x (P_x) is fixed then (6) is the Engel curve for the good x.

$$\boldsymbol{y} = \frac{\alpha}{\alpha+\beta} \cdot \frac{\boldsymbol{m}}{P_y} \textbf{(7)} \rightarrow \text{(This is the demand function for the y)}$$

Note: When the price of the good y (P_y) is fixed then (7) is the Engel curve for the good y.

Re-writing (6 & 7) as $P_x = \frac{\alpha}{\alpha+\beta} \cdot \frac{\boldsymbol{m}}{x}$ & $P_y = \frac{\alpha}{\alpha+\beta} \cdot \frac{\boldsymbol{m}}{y}$

gives the inverse demand functions.

Homogeneous Functions

A function is said to be homogeneous of degree r, if multiplication of each of its independent variables by a constant (j) will alter the value of the function by the proportion j^r, that is, if:

$$f(jx_1, \ldots\ldots, jx_2) = j^r f(x_1, \ldots\ldots, x_2)$$

For example, a homogeneous function of two variables x and y is a real-valued function that satisfies the condition $f(\alpha x, \alpha y) = \alpha^k f(x, y)$ for some constant k and all real numbers α. The constant k is called the degree of homogeneity.

Steps to identify the degree of homogeneity:

To be Homogeneous a function must pass this test:

1. **Homogeneous** is when we can take a function: f(x,y)
2. multiply each variable by t:f(tx,ty)
3. **and then** can rearrange it to get this :t^kf(x,y)
4. **Decide the degree of homogeneity, which is the value of K.**

Example 1

The function $f(x, y) = x + 3y$ is homogeneous of degree 1.

Solution:

Step 1: The function is $f(x, y) = x + 3y$

Step 2: Multiplying each variable by t, we get: $f(tx, ty) = tx + 3ty$

Step 3: Rearrange it by factoring out t, we get: $\mathrm{tf(x, y)} = \mathrm{t(x + 3y)}$

Step 4: Decision: the function $f(x, y) = \mathrm{x} + 3\mathrm{y}$ is homogeneous of degree 1.

Example 2

The function $f(x, y) = x^2 + y^2$ is homogeneous of degree 2.

Solution:

Step 1: The function is $f(x, y) = x^2 + y^2$

Step 2: Multiplying each variable by t, we get:

$$f(tx, ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2$$

Step 3: Rearrange it by factoring out t, we get: $\mathrm{t}^k\mathrm{f(x, y)} = \mathrm{t}^2(x^2 + y^2)$

Step 4: Decision: the function $f(x, y) = x^2 + y^2$ is homogeneous of degree 2.

Example 3

The function $f(x, y) = 4x^2 + y^2$ is homogeneous of degree 2.

Step 1: The function is $f(x, y) = 4x^2 + y^2$

Step 2: Multiplying each variable by t, we get:

$$f(tx, ty) = 4(tx)^2 + (ty)^2 = 4t^2x^2 + t^2y^2$$

Step 3: Rearrange it by factoring out t, we get: $\mathrm{t^k f(x, y)} = \mathrm{t}^2(4x^2 + y^2)$

Step 4: Decision: the function $f(x, y) = 4x^2 + y^2$ is homogeneous of degree 2.

Example 4

The function $Q = f(K, L) = AK^{\alpha} L^{\beta}$ is homogeneous of degree $\alpha + \beta$.

Step 1: The function is $f(L, K) = AK^{\alpha} L^{\beta}$

Step 2: Multiplying each variable by t, we get:

$$f(tK, tL) = A(tK)^{\alpha}(tL)^{\beta} = At^{\alpha}K^{\alpha}t^{\beta}L^{\beta}$$

Step 3: Rearrange it by factoring out t, we get: $\mathrm{t^k f}(K, L) = \mathrm{t}^{\alpha+\beta}(\mathrm{A}K^{\alpha}L^{\beta})$

Step 4: Decision: the function$Q = f(K, L) = AK^{\alpha} L^{\beta}$ is homogeneous of degree $\alpha + \beta$.

Example 5

If you have the following polynomial function $f(\mathrm{x, y}) = \mathrm{x}^3 + \mathrm{y}^2$ is **NOT** homogeneous.

Step 1: The function is $f(x, y) = \mathrm{x}^3 + \mathrm{y}^2$

Step 2: Multiplying each variable by t, we get:

$$f(tx, ty) = (tx)^3 + (ty)^2 = t^3x^2 + t^2y^2$$

Step 3: Rearrange it by factoring out t, we get: $f(tx, ty) = \mathrm{t}^2(\mathrm{t}x^3 + y^2)$

Step 4: Decision: since $\mathrm{t}x^3 + y^2 \neq f(x, y) = \mathrm{x}^3 + \mathrm{y}^2$, the function $f(x, y) = \mathrm{x}^3 + \mathrm{y}^2$ is **NOT** homogeneous.

What is Linear Homogeneous Production Function?

The **Linear Homogeneous Production Function (**homogeneous functions of the first degree) implies that with the proportionate change in all the factors of production, the output also increases in the same proportion. Such as, if the input factors are doubled the output also gets doubled. This is also known as **constant returns to a scale.**

Note: a function which is homogeneous of the first degree is not necessarily linear in itself.

Example: the common linearly homogeneous production function of the form:

$$\boldsymbol{Q = f(K, L)}$$

What unique properties characterize this linearly homogeneous production function?

> **Property 1**: the average physical product of labor (APP_L) and the average physical product of capital (APP_K) can be expressed as functions of the capital-labor ratio, k = K/ L, alone.
>
> **Proof**:

If we assume $\boldsymbol{j = \frac{1}{L}}$, then by multiplying each input factor in the production function by $\boldsymbol{j}$, we get: $\boldsymbol{\mathrm{j}Q = \mathrm{j}f(K, L)}$, that is

$$\boldsymbol{f(\frac{K}{L}, \frac{L}{L}) = f(\frac{K}{L}, 1) = f(k, 1)} \,,$$

Note: the right side in effect becomes a function of the capital-labor ratio k alone (say, $\emptyset(\boldsymbol{k})$), which is a function with a single argument, $\boldsymbol{k}$.

That is,

$$APP_L = \frac{Q}{L} = \emptyset(\boldsymbol{k})$$

By the same token,

$$APP_K = \frac{Q}{K} = \frac{Q}{L} . \frac{L}{K} = \frac{\emptyset(\boldsymbol{k})}{\boldsymbol{k}}$$

Since both average products depend on $\boldsymbol{k}$ alone, linear homogeneity implies that, as long as the K/L ratio is kept constant, the average products will be constant.

Therefore, while the production function is homogeneous of degree **one**. both **APPL** and **APPK** are homogeneous of degree **zero** in the variables K and L , since equal proportionate changes in K and L (maintaining a constant k) will **not** alter the magnitudes of the average products.

Note: a function is homogeneous of degree **zero when:**

$$\boldsymbol{f(tx, ty) = t^0 f(x, y) = f(x, y)}$$

Property 2: The marginal physical products MPP_L and MPP_K can be expressed as functions of $\boldsymbol{k}$ alone.

Proof:

To find the marginal products, we first write the total product as:

$$Q = L.\emptyset(\boldsymbol{k})$$

and then differentiate Q with respect to K and L.

that is,

$$MPP_K = \frac{\partial Q}{\partial K} = \frac{\partial}{\partial K}[L.\emptyset(\boldsymbol{k})] \text{ \textbf{[\underline{chain rule}]}}$$

$$= \boldsymbol{L}\frac{\partial \emptyset(\boldsymbol{k})}{\partial \boldsymbol{k}}.\frac{\partial \boldsymbol{k}}{\partial K} = L.\emptyset'(\boldsymbol{k})\left(\frac{1}{L}\right) = \emptyset'(\boldsymbol{k})$$

Note that: $.\frac{\partial \boldsymbol{k}}{\partial \mathrm{K}} = \frac{\partial}{\partial \mathrm{K}}\left(\frac{K}{L}\right) = \left(\frac{1}{L}\right)$

By the same token,

$$MPP_L = \frac{\partial Q}{\partial L} = \frac{\partial}{\partial L}[L.\emptyset(\boldsymbol{k})]$$

$$= \emptyset(\boldsymbol{k}) + \mathrm{L}.\frac{\partial \emptyset(\boldsymbol{k})}{\partial L} \qquad \text{\textbf{[\underline{product rule}]}}$$

$$= \emptyset(\boldsymbol{k}) + \mathrm{L}.\emptyset'(\boldsymbol{k}).\frac{\partial \boldsymbol{k}}{\partial L} \quad \text{\textbf{[\underline{chain rule}]}}$$

$$= \emptyset(\boldsymbol{k}) + \mathrm{L}.\emptyset'(\boldsymbol{k}).\left(\frac{-K}{L^2}\right)$$

$$= \emptyset(\boldsymbol{k}) - \boldsymbol{k}\,\emptyset'(\boldsymbol{k})$$

Note that: $.\frac{\partial \boldsymbol{k}}{\partial \mathrm{L}} = \frac{\partial}{\partial \mathrm{L}}\left(\frac{K}{L}\right) = \frac{-K}{L^2}$

The above proofs indeed show that MPP_K and MPP_L are functions of $\boldsymbol{k}$ alone.

Note: Like average products, the marginal products will remain the same as long as the capital-labor ratio is held constant; they are homogeneous of degree zero in the variables K and L.

Property 3 (Euler's theorem):

If Q = f (K , L) is linearly homogeneous, then

$$K\,\frac{\partial \boldsymbol{Q}}{\partial \mathrm{K}} + L\,\frac{\partial \boldsymbol{Q}}{\partial \mathrm{L}} = Q$$

Proof:

$$K\,\frac{\partial \boldsymbol{Q}}{\partial \mathrm{K}} + L\,\frac{\partial \boldsymbol{Q}}{\partial \mathrm{L}} = K\emptyset'(\boldsymbol{k}) + \mathrm{L}[\emptyset(\boldsymbol{k}) - \boldsymbol{k}\,\emptyset'(\boldsymbol{k})]$$

$$= K\emptyset'(\boldsymbol{k}) + \mathrm{L}\emptyset(\boldsymbol{k}) - \boldsymbol{k}\,\emptyset'(\boldsymbol{k})$$

$$= \mathrm{L}\emptyset(\boldsymbol{k}) = \mathrm{Q}$$

Remember that:

- ✓ $APP_L = \frac{Q}{L} = \emptyset(\boldsymbol{k})$
- ✓ $MPP_L = \frac{\partial Q}{\partial L} = \emptyset(\boldsymbol{k}) - \boldsymbol{k}\,\emptyset'(\boldsymbol{k})$

Notes:

The previous property is valid for *any* values of K and L.

The interpretation of the previous property: the value of a linearly homogeneous function can always be expressed as a sum of terms, each of which is the product of one of the independent variables and the first-order partial derivative with respect to that variable, regardless of the levels of the two inputs actually employed.

Economically, this property means that under conditions of constant returns to scale, if each input factor is paid the amount of its **marginal product**, the total product will be exactly exhausted by the distributive shares for all the input factors, or, equivalently, the pure economic profit will be zero.

Note that the previous situation is descriptive of the long-run equilibrium under pure competition. The zero economic profit in the long-run equilibrium is brought about by the forces of competition through the entry and exit of firms, regardless of the specific nature of the production functions actually prevailing. Thus it is not mandatory to have a production function that ensures product exhaustion for any and all (K, L) pairs. Moreover. when imperfect competition exists in the factor markets, the remuneration to the factors may not be equal to the marginal products, and, consequently, Euler's theorem becomes irrelevant to the distribution picture.

Production cost minimization under output constraint (Cobb-Douglas Production Function

One specific production function widely used in economic analysis is the Cobb Douglas production function.

- **A generalized version of this function**

$$Q = AK^{\alpha} L^{\beta}$$

where A is a positive constant. and α & β is a positive fraction.

Features of the previous function:

1. It is homogeneous of degree $\alpha + \beta$.
2. If $\alpha + \beta = 1$, it means the function is linearly homogeneous.
3. Its isoquants are negatively sloped throughout and strictly convex for positive values of K and L.
4. It is strictly quasiconcave for positive K and L.
5. The previous function is not linear (but it may be linearly homogeneous).
6. Its isoquants have negative slopes and strictly convexity, which can be verified by the sign of the first and second partial derivatives.
7. The production exhibits **increasing returns to scale** if the production function is homogeneous of a degree greater than one.
8. The production exhibits **constant returns to scale** if the production function is linearly homogeneous.
9. The production exhibits **decreasing returns to scale** if the production function is homogeneous of a degree less than one.

To sum-up: in the General form of Cobb-Douglas production function $\boldsymbol{Q\ (K, L)\ =\ A\ K^{\alpha} \alpha\ L^{\beta}}$

- ✓ $\boldsymbol{\alpha + \beta = 1}$ (constant returns to scale)
- ✓ $\boldsymbol{\alpha + \beta > 1}$ (increasing returns to scale)
- ✓ $\boldsymbol{\alpha + \beta < 1}$ (decreasing returns to scale)
- **A special case of Cobb-Douglas production function is to be linearly homogeneous as follows:**

$$Q\ (K, L)\ =\ A\ K^{\alpha} L^{1-\alpha}$$

- **What is the economic interpretation of (α) and (1-α)?**

The exponent of each input variable indicates the relative share of that input in the total product. That is, if each input is assumed to be paid by the amount of its marginal product,

(1) the relative share of total product accruing to capital will be:

$$\frac{K(\partial Q/\partial K)}{Q} = \frac{KA\alpha K^{\alpha-1}}{LAK^{\alpha}} = \alpha$$

(2) Similarly, labor's relative share will be

$$\frac{L(\partial Q/\partial L)}{Q} = \frac{LA(1-\alpha)K^{\alpha}}{LAK^{\alpha}} = 1-\alpha$$

Note that:

We can also interpret the exponent of each input variable as the partial elasticity of output with respect to that input. Since, $\frac{K(\partial Q/\partial K)}{Q} = \frac{\partial Q/\partial K}{Q/K} = \varepsilon_{QK}$ and $\frac{L(\partial Q/\partial L)}{Q} = \frac{\partial Q/\partial L}{Q/L} = \varepsilon_{QL}$

- **What about the meaning of the constant A?**

"A" may be considered as an efficiency parameter , i.e., as an indicator of the state of technology. Note that, for given values of K and L , the magnitude of A will proportionately affect the level of Q.